On the $\mathbb{A}^1$-Euler characteristic of the variety of maximal tori

k - field, char $k=0$ (for simplicity)

$X \in Sm_k \rightsquigarrow \chi^{\mathbb{A}^1}(X) \in GW(k)$ - Grothendieck-Witt ring of sym. bil. forms$/k$
= group completion of the additive monoid of sym. bil. forms (quadratic)

Def $(\mathcal{C}, \otimes, 1)$ - symm. monoidal cat.

- $X \in \mathcal{C}$ is strongly dualizable if $\exists X^\vee \in \mathcal{C}$ and $coev: 1 \to X \otimes X^\vee$, $ev: X^\vee \otimes X \to 1$

such that:

$$\underbrace{X \simeq 1 \otimes X \xrightarrow{coev \otimes id} X \otimes X^\vee \otimes X \xrightarrow{id \otimes ev} X \otimes 1 \simeq X}_{id} \text{ is identity map}$$

$$\underbrace{X^\vee \simeq X^\vee \otimes 1 \xrightarrow{id \otimes coev} X^\vee \otimes X \otimes X^\vee \xrightarrow{ev \otimes id} 1 \otimes X^\vee \simeq X^\vee}_{id} \text{ is identity map}$$

- if $X \in \mathcal{C}$ is strongly dualizable then

$$\underbrace{1 \xrightarrow{coev} X \otimes X^\vee \xrightarrow{tw} X^\vee \otimes X \xrightarrow{ev} 1}_{=: \chi(X)} \text{ - categorical Euler characteristic of } X$$

$\chi(X) \in End_{\mathcal{C}}(1)$

Ex: $(Vect_k, \otimes, k)$

$coev: k \to V \otimes V^\vee$ - "scalar matrices"

V-strongly dual $\Leftrightarrow \dim V < \infty$ $\quad ev: V^\vee \otimes V \to k$ - trace

$\rightsquigarrow \chi(V) = \dim V \in End_k(k) \simeq k$

Rm. $\mathcal{C}$-sym. monoidal & triangulated

1) $\chi(X \otimes Y) = \chi(X) \cdot \chi(Y)$
2) $\chi(Y) = \chi(X) + \chi(Z)$ if $X \to Y \to Z \to X[1]$ triangle.
   in particular, $\chi(X[1]) = -\chi(X)$

Ex: • $\mathcal{C} = D(k)$ $\chi(C_\bullet) = \sum_i (-1)^i \dim_k H_i(C_\bullet)$

$SH \to D(\mathbb{Q})$

$End(\mathbb{S}) \hookrightarrow End_{\mathbb{Q}}(\mathbb{Q})$, $End(\mathbb{S}) \simeq \mathbb{Z}$, $End_{\mathbb{Q}}(\mathbb{Q}) \simeq \mathbb{Q}$

• $\mathcal{C} = SH$, X-finite CW-complex $\Rightarrow \chi(\Sigma^\infty X_+) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(X, \mathbb{Q}) = \chi^{top}(X)$

Motivic homotopy theory

presheaves (i.e. contravariant functors)

Def. $\mathrm{PreSh}(Sm_k, Spc_\bullet) \to \mathrm{PreSh}(Sm_k, Spt)$ — spectra (topology)

Sm_k ↗ representable discrete presheaf — pointed spaces

identify spectra up to hom. w.e. — impose Nisnevich descent

$$SH^{S^1}(k) := \mathrm{PreSh}(Sm_k, Spt)\left[\text{homotopy w.e.}^{-1}, \mathrm{Nis}^{-1}, \mathbb{A}^1 \simeq \mathrm{Spec}\, k\right]$$

– triangulated sym. monoidal — $\mathbb{A}^1$-invariance

$$SH(k) = SH^{S^1}(k)\left[(\otimes \mathbb{P}^1)^{-1}\right]$$

Rk: • Chow groups, motivic coh, Quillen K-thy, $H^*_{\acute{e}t}(-, \mu_n^{\otimes *})$, Hermitian K-thy, alg cobordism, ... are representable in $SH(k)$

• $SH(k)$ – triangulated sym. monoidal, $\mathrm{End}_{SH(k)}(1) \simeq GW(k)$ (Morel '12)

$\uparrow \Sigma^\infty(\ )_+$ $\quad$ $\mathbb{P}^1 \to \mathbb{P}^1 \longleftrightarrow \langle u \rangle$

Sm_k $\quad$ $[x:y] \mapsto [ux:y]$

needs char $k = 0$

$X \in Sm_k \Rightarrow \Sigma^\infty X_+$ is strongly dual (Riou '05)

$$\chi^{\mathbb{A}^1}(X) := \chi^{SH(k)}(\Sigma^\infty X_+) \in GW(k)$$

← $\mathbb{A}^1$-Euler characteristic

Ex: • $k = \mathbb{C}$

$$GW(\mathbb{C}) \overset{rk}{\simeq} \mathbb{Z}$$
$$\chi^{\mathbb{A}^1}(X) = \chi^{top}(X(\mathbb{C}))$$

$$SH(\mathbb{C}) \xrightarrow{\mathrm{Betti}_{\mathbb{C}}} SH$$
$$\mathbb{Z} \simeq \mathrm{End}(1) \xrightarrow{\sim} \mathrm{End}(1) \simeq \mathbb{Z}$$

• $k = \mathbb{R}$

$$GW(\mathbb{R}) \xhookrightarrow[(rk,\ sign)]{} \mathbb{Z} \times \mathbb{Z}$$
$$rk\ \chi^{\mathbb{A}^1}(X) = \chi^{top}(X(\mathbb{C}))$$
$$sign\ \chi^{\mathbb{A}^1}(X) = \chi^{top}(X(\mathbb{R}))$$

$$SH(\mathbb{R}) \xrightarrow{\mathrm{Betti}_{\mathbb{C}}} SH$$
$$GW(\mathbb{R}) \xrightarrow{rk} \mathbb{Z}$$
$$SH(\mathbb{R}) \xrightarrow{\mathrm{Betti}_{\mathbb{R}}} SH$$
$$GW(\mathbb{R}) \xrightarrow{sign} \mathbb{Z}$$

• k - arbitrary

1) X - projective, $\dim X = d$

$$H^i(X,\Omega^j) \times H^{d-i}(X,\Omega^{d-j}) \to H^d(X,\Omega^d) \simeq k \qquad (\mathrm{Hdg}, \mathrm{Tr})$$

$\rightsquigarrow$ sym. bil. form φ on $\bigoplus H^i(X,\Omega^j)$ $\quad \text{''}$

Levine–Raksit '18: $\chi^{\mathbb{A}^1}(X) = \langle \bigoplus_{i+j \text{ even}} H^i(X,\Omega^j), \varphi \rangle - \langle \bigoplus_{i+j \text{ odd}} H^i(X,\Omega^j), \varphi \rangle$

2) X - projective hypersurface $\{f=0\}$ $\rightsquigarrow$ one may interpret $(\mathrm{Hdg}, \mathrm{Tr})$ in terms of

Levine–Lehalleur–Srinivas '21 $\quad \mathrm{Jac}(X) = k[x_0, \ldots x_n] / \left(\frac{\partial f}{\partial x_i}\right)$

E.g. for $f = a_0 x_0^{2m} + \ldots + a_{2n+1} x_{2n+1}^{2m}$ $\quad \chi^{\mathbb{A}^1}(X) = ? \cdot (\langle 1 \rangle + \langle -1 \rangle) + \langle 2m \rangle + \langle -2m \cdot \prod a_i \rangle$

3) X - proj $\rightsquigarrow$ in terms of $HH_{\bullet}(X)$

ABOWZ '20

<u>Question?</u> How to compute $\chi(X)$ for $X = \mathrm{Spec}\, R$?

<u>Rk:</u> $K_0(\mathrm{Var}) = \left(\bigoplus_{X \in Sm_k} \mathbb{Z}[X]\right) / [X] = [Z] + [X - Z]$ for $Z \hookrightarrow X$ closed

$$\chi_c^{\mathbb{A}^1}(X) := \langle -1 \rangle^{\dim X} \cdot \chi^{\mathbb{A}^1}(X).$$

$$\begin{tikzcd} & Sm_k \arrow[dl] \arrow[dr, "\chi_c^{\mathbb{A}^1}"] & \\ K_0(\mathrm{Var}) \arrow[rr, "\exists !"] & & GW(k) \end{tikzcd}$$

<u>Thm</u> (Levine) $X \in Sm_k$, $f: Y \to X$ - Nisnevich locally trivial fibration with fiber V. Suppose that $\chi^{\mathbb{A}^1}(V)$ is invertible (in $GW(k)$)

$\Rightarrow$ for every coh. th. E^* repr. in $SH(k)$ $f^*: E^*(X) \to E^*(Y)$ is split injective

Corollary: $X \in Sm_k$, $G \in Sm_k$ - alg. group, $N \leq G$, $\mathcal{G}/X$ - Nis. loc. trivial G-torsor. Suppose that $\chi^{\mathbb{A}^1}(G/N)$ is invertible

$\Rightarrow \exists\, f: Y \to X$ such that 1) $f^*: E^*(X) \to E^*(Y)$ is split inj.

$\mathcal{G}/N$ $\qquad$ 2) $f^*\mathcal{G}$ is induced from an N-torsor

<u>Ex:</u> $G = GL_n$, $N = N_{GL_n}(T)$ - monomial matrices

GL_n - torsor $\leftrightarrow$ rank n vector bundle $/X$

N - torsor $\leftrightarrow$ " direct sum of n line bundles on S_n-Galois cover,

"direct sum of line bundles up to permutation"

<u>Thm</u> (A. '20) G - reductive group $/k$, T - maximal torus, $N = N_G(T)$.

$\Longrightarrow \chi^{\mathbb{A}^1}(G/N)$ is invertible

Sketch of the plan of the proof:

$H^*((G/N)_{\bar{k}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$

1) $\varphi \in GW(k)$ is invertible $\Longleftrightarrow$ $\mathrm{rk}\,\varphi = \pm 1$

sign $\varphi = \pm 1$ for all signatures

orderings of k

2) For sign reduce to $k = \mathbb{R}$ (semisimple groups over real closed fields)

3) Compute $\chi^{top}((G/N)(\mathbb{R}))$:

$$G(\mathbb{R}) \circlearrowright (G/N)(\mathbb{R}) = \{\tilde{T} \leq G \mid \tilde{T} \text{ - maximal torus}\} \overset{conj}{\circlearrowleft} G(\mathbb{R})$$

$T_1, \ldots T_n \leq G$ - pairwise non-conj. max. tori $N_i := N_G(T_i)$

$$\Rightarrow (G/N)(\mathbb{R}) = \coprod_i G(\mathbb{R})/N_i(\mathbb{R})$$

$\mathrm{crk}\, T_i :=$ dimension of maximal compact torus in $T_i(\mathbb{R})$

$\mathrm{crk}\, G := \max \mathrm{crk}\, T_i$

- $\exists!\, i$ s.t. $\mathrm{crk}\, T_i = \mathrm{crk}\, G$; in this case $\chi^{top}(G(\mathbb{R})/N_i(\mathbb{R})) = 1$
- if $\mathrm{crk}\, T_i < \mathrm{crk}\, G$ then $\chi^{top}(G(\mathbb{R})/N_i(\mathbb{R})) = 0$

$$\begin{tikzcd}
G(\mathbb{R})/T_i(\mathbb{R}) \arrow[d, "\text{finite cover}"'] & G(\mathbb{R})/(S^1)^{\times \mathrm{crk}\, T_i} \arrow[l, "fin"'] \arrow[d, "\text{fibration with fiber } (S^1)^{\times(\mathrm{crk}\, G - \mathrm{crk}\, T_i)}"] \\
G(\mathbb{R})/N_i(\mathbb{R}) & G(\mathbb{R})/(S^1)^{\times \mathrm{crk}\, G}
\end{tikzcd}$$

$\chi(S^1) = 0 \Rightarrow 0$