23/11/2023
Mainz

Combing a hedgehog over a field.

jt. with Marc Levine.

<u>Topology</u>: $S^2 = \{x^2+y^2+z^2=1\} \Rightarrow T_{S^2}$ does not have a non-vanishing section (= nowhere vanishing)

$S^2_{\mathbb{C}} = \{x^2+y^2+z^2=1\} \subseteq \mathbb{C}^3 \Rightarrow T_{S^2_{\mathbb{C}}}$ has a non-van. section

- open manifold, $\dim = 4$, $\mathrm{rk}_{\mathbb{R}} T_{S^2_{\mathbb{C}}} = 4 \Rightarrow \exists$ non-van. continous section $\xrightarrow{\text{Grauert-Oka principle}} \exists$ non-van. analytic section
- Murthy: $\underbrace{c_2(T_{S^2_{\mathbb{C}}})}_{=0} \in \underbrace{CH^2(S^2_{\mathbb{C}})}_{=0} \Rightarrow \exists$ non-vanish. algebraic section
- easy to write down an explicit non-van. section.

<u>Question</u>: For which k does $T_{S^2_k}$ have a non-vanish. section? $S^2_k = \{x^2+y^2+z^2=1\} \subseteq \mathbb{A}^3_k$

More generally, char $k \neq 2$, $Q^0 = \{q=1\} \subseteq \mathbb{A}^{n+1}_k$, q regular quadratic form, does T_{Q^0} have a non-van. sect.?

<u>In particular</u>, does $T_{S^2_{\mathbb{Q}_2}}$ has a non-vanishing section? (Umberto Zannier) ($\mathbb{Q}_2$ = dyadic numbers)

<u>Explicits sections</u>:

$q = a_1x_1^2 + \dots + a_{n+1}x_{n+1}^2$, $a_i \in k^* \rightsquigarrow$

$$0 \to T_{Q^0} \to T_{\mathbb{A}^{n+1}}|_{Q^0} \to N_{Q^0/\mathbb{A}^{n+1}} \to 0$$

$$T_{\mathbb{A}^{n+1}}|_{Q^0} \cong \mathcal{O}_{Q^0}^{\oplus n+1} \to N_{Q^0/\mathbb{A}^{n+1}} \cong \mathcal{O}_{Q^0} \to 0$$

$(s_1, s_2, \dots s_{n+1}) \mapsto 2a_1x_1s_1 + \dots + 2a_nx_ns_n$ $\quad \langle \nabla q, (s_1 \dots s_n)\rangle = 0$

Non-vanishing sections $\leftrightsquigarrow \{s_1, \dots, s_{n+1}\}$ - regular functions on Q^0 s.t.

- $\sum a_ix_is_i = 0$
- no common zeroes of s_i on Q^0

① n is odd: $(-a_2x_2, a_1x_1, -a_4x_4, a_3x_3, \dots, -a_{n+1}x_{n+1}, a_nx_n)$

② n is even, q is isotropic, i.e. $q=0$ has a solution in k $\rightsquigarrow$ after change of basis $q = 2x_1x_2 + b_3x_3^2 + \dots + b_{n+1}x_{n+1}^2$

$$0 \to T_{Q^0} \to \mathcal{O}_{Q^0}^{\oplus n+1} \to \mathcal{O}_{Q^0} \to 0$$

$(s_1, \dots, s_{n+1}) \mapsto 2x_2s_1 + 2x_1s_2 + \sum 2b_ix_is_i$

section: $(0, -b_3x_3, x_1, -b_5x_5, b_4x_4, \dots, -b_{n+1}x_{n+1}, b_nx_n)$.

common zeroes: $0 = x_1 = x_3 = x_4 = \dots = x_{n+1}$ & $2x_1x_2 + \sum b_ix_i^2 = 0$ $\Rightarrow \emptyset$

Topology: $e(T_{S^2}) \in H^2_{Sing}(S^2) \simeq \mathbb{Z}$

Gauss-Bonnet $\longrightarrow$ $\parallel$

$\chi(S^2) = 2$ $\Rightarrow$ T_{S^2} does not have a non-van. section.

Alg. geometry (motivic homotopy theory):

Barge-Morel '00:

local system

$\widetilde{CH}^n(X, L)$ - Chow-Witt groups $\rightsquigarrow H^n_{Sing}(M, \mathbb{Z}(\mathcal{L}))$

X smooth$/k$, L line bundle$/X$

for $n = \dim X$, $L = \omega_X$:

$\widetilde{CH}_0(X) := \widetilde{CH}^n(X, \omega_X) = \mathrm{Coker}(\oplus ? \to \bigoplus_{x \in X^{(n)}} GW(k(x)))$

Grothendieck-Witt group of reg. quadr. forms$/k(x)$

E/X - rank n vector bundle

$\rightsquigarrow$ $e(E) \in \widetilde{CH}^n(X; \det E^\vee) \rightsquigarrow e(E) \in H^n_{Sing}(M, \mathbb{Z}(\det E^\vee))$

Thm (Morel '12 + Asok-Hoyois-Wendt '17 + Asok-Fasel '16) k - perfect field, X/k - smooth affine, E/X - vector bundle, $\operatorname{rank} E = \dim X$, $\det E \simeq \mathcal{O}_X$ $\Rightarrow$

$e(E) = 0 \iff E$ has a non-vanishing section.

Rem: $\det T_{Q^0} \simeq \mathcal{O}_{Q^0}$ from the exact sequence.

Question: when $e(T_{Q^0}) = 0$?

Motivic Gauss-Bonnet theorem:

X - smooth proper$/k$ $\rightsquigarrow$ $\deg_{GW}: \widetilde{CH}_0(X) \to GW(k)$

Thm (Levine-Raksit '20, Déglise-Jin-Khan '21) $\deg_{GW}(e(T_X)) = \chi^{\mathbb{A}^1}(X)$ - $\mathbb{A}^1$-Euler characteristic, "computable" e.g. via Hodge cohomology.

$Q := \{\sum a_i x_i^2 = x_0^2\} \subseteq \mathbb{P}^{n+1}$

$Q^\infty := \{\sum a_i x_i^2 = 0\} \subseteq \mathbb{P}^n$ $\rightsquigarrow$ $Q^\infty \xrightarrow{i} Q \xhookleftarrow{j} Q^0$

$$\begin{tikzcd}
\widetilde{CH}_0(Q^\infty) \arrow[r] \arrow[dr, "\deg_{GW}"'] & \widetilde{CH}_0(Q) \arrow[r] \arrow[d, "\deg_{GW}"] & \widetilde{CH}_0(Q^0) \\
& GW(k) &
\end{tikzcd}$$

$$\begin{tikzcd}
e(T_Q) \arrow[r, mapsto] \arrow[d, mapsto] & e(T_{Q^0}) \\
\chi^{\mathbb{A}^1}(Q) &
\end{tikzcd}$$

$\langle a,b\rangle := a y_1^2 + b y_2^2 \in GW(k)$

Thm $n>0$ even, k - perfect field, char $k \neq 2$
(A.-Levine '23) ① Suppose Q_0 has a rational point. than T_{Q^0} has a non-van. section $\iff \langle 1, \prod a_i\rangle \in \deg_{GW}(\widetilde{CH}_0(Q^\infty))$

② T_{Q^0} has a non-van. section $\Rightarrow \langle 1, \prod a_i\rangle \in \deg_{GW}(\widetilde{CH}_0(Q^\infty))$

Pf: above, for ① use that $\widetilde{CH}_0(-)$ is a stable birational invariant (Feld '22),

whence $\deg_{GW}: \widetilde{CH}_0(Q) \to GW(k)$ - iso.

Def. q - quadratic form $/k$, $D(q) \subseteq k^\times$ - set of non-zero values,
$D(q)^2 = \{a\cdot b \mid a,b \in D(q)\}$, $[D(q)]$, $[D(q)^2]$ - subgroups of $k^\times$ generated by respective sets.

Thm $n>0$ even, k - perfect field, char $k \neq 2$
(A.-Levine) $q = \sum_{i=1}^{n+1} a_i x_i^2$, $Q^0 = \{q = 1\} \subseteq \mathbb{A}^{n+1}_k$

① $Q^0(k) \neq \emptyset$. Then T_{Q^0} has a non-van. section $\iff -1 \in [D(q)]$

② T_{Q^0} has a non-van. section $\Rightarrow -\prod a_i \in [D(q)^2]$

Lm. Q^∞/k - smooth proj quadric given by $q=0$. $\Rightarrow \langle a,b\rangle \in \deg_{GW}(\widetilde{CH}_0(Q^\infty)) \iff ab \in [D(q)]$

Pf: uses explicit computations with Scharlau's transfers & Knebusch norm principle.

Ex: $S^2_{\mathbb{Q}_2}$. $\mathbb{Q}_2/\mathbb{Q}_2^2 = \left\{ \begin{matrix} 1, 3, 5, 7 \\ 2, 6, 10, 14 \end{matrix} \right\}$, $7 = -1$, $D(x^2+y^2+z^2) = \{1,3,5,2,6,10,14\}$
$-1 \in [D(x^2+y^2+z^2)]$.

$\Rightarrow T_{S^2_{\mathbb{Q}_2}}$ has a non-vanishing section.

Question: Explicit formula?

Corollary: $S^n_k = \{x_1^2 + \dots + x_{n+1}^2 = 1\} \subseteq \mathbb{A}^{n+1}$. $T_{S^n_k}$ has a non-van. section
$\iff$ ① n is odd
② $n>0$ is even & $y_1^2 + \dots + y_{2n+1}^2 = -1$ has a solution in k
$s(k) \leq 2n$

Rem $s(k)$ = minimal N s.t. $y_1^2 + \dots + y_N^2 = -1$ has a solution. level of k. Pfister: $s(k) = \infty$ or $s(k) = 2^m$.

In particular, S^n_k, $n>0$, has a non-vanishing vector field if:
- char $k = p > 2$
- $\mathbb{Q}_p \subseteq k$
- k is purely imaginary number field