Toric varieties.

LMU
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I. Introduction.

Algebraic geometry: studies zero sets $\begin{cases} f_1(x_1,\dots,x_n)=0 \\ \dots \\ f_m(x_1,\dots x_n)=0 \end{cases}$ in $\mathbb{C}^n$, $f_i\in\mathbb{C}[x_1,\dots x_n]$
(affine alg. varieties)

or $\begin{cases} f_1(x_0,\dots,x_n)=0 \\ \dots \\ f_m(x_0,\dots x_n)=0 \end{cases}$ in $\mathbb{P}^n_{\mathbb{C}}$, $f_i\in\mathbb{C}[x_0,\dots x_n]$ homogeneous
(proj. alg. varieties)

Hard for general $\{f_i\}$, e.g. $f\in\mathbb{C}[x_1,\dots x_5]$, $\deg f=3$, general.
Does $\{f=0\}\subseteq\mathbb{C}^5$ have a rational parametrization? I.e. $\{p_1,\dots p_5\}\in\mathbb{C}(t_1,\dots t_4)$
s.t. $f(p_1,\dots p_5)=0$ & $(p_1,\dots p_5): \underset{\substack{\mathbb{C}^5\text{ (zariski)}\\ \text{open, non-empty}}}{U} \longrightarrow \{f=0\}$ is injective? for some U.

(rational parametrization of a circle:
$\{x^2+y^2=1\}\subseteq\mathbb{C}^2$, $p_1(t)=\frac{2t}{1+t^2}$, $p_2(t)=\frac{1-t^2}{1+t^2}$)

Plenty of general theory, but some specific classes of varieties are much better understood, e.g. ones parametrizing some geometric objects ("moduli spaces")

Ex: $\mathbb{P}^n_{\mathbb{C}}=\{\text{linear subspaces}\subseteq\mathbb{C}^{n+1}\}$
$Gr_{\mathbb{C}}(k,n)=\{V\subseteq\mathbb{C}^n \mid \dim_{\mathbb{C}}V=k\}$
- have cellular decomposition $(=\sqcup\,\mathbb{C}^{n_i})$
- Homogeneous under a matrix group, in particular $\mathrm{Aut}(X)$ is big

There is only a fin. number of varieties proj. homogeneous under a matrix group in each dimension.

Def Toric variety = alg. variety X with an open dense subset $(\mathbb{C}^*)^n\subseteq X$ such that the action $(\mathbb{C}^*)^n\times(\mathbb{C}^*)^n\to(\mathbb{C}^*)^n$ extends to an action $(\mathbb{C}^*)^n\times X\to X$.
"toroidal embeddings" ~ toric varieties.

Rk. systematically introduced by Demazure '1970 (over Spec $\mathbb{Z}$!) in his work on Cremona group.

Ex: 1) $(\mathbb{C}^*)^n$, $\mathbb{C}^n$, $\mathbb{P}^n$ with the action $(t_1,\dots t_n)\cdot[x_0:\dots:x_n]=[t_1x_0:t_2x_1:\dots:t_nx_{n-1}:x_n]$
2) $\{x^3-y^2=0\}\subseteq\mathbb{C}^2$, $t\cdot(x,y)=(t^2\cdot x, t^3\cdot y)$, $\{x^3-y^2=0\}\setminus(0,0)\simeq\mathbb{C}^*$
3) $\{xy-zw=0\}\subseteq\mathbb{C}^4$, $(t_1,t_2,t_3)\cdot(x,y,z,w)=(t_1x, t_2y, t_3z, t_1t_2t_3^{-1}w)$

(Normal) toric varieties admit a combinatorial description via "fans" (certain families of cones in lattices), e.g. $\mathbb{P}^2 \rightsquigarrow$  $\leq \mathbb{Z}^2$, and one can read off geometric properties of X from the fan.

Applications to: physics (mirror symmetry), study of polytopes, topology, solving polynomial equations, etc.

<u>Ex</u>: $f = -7x - 9y - 10x^2 + 17xy + 10y^2 + 16x^2y - 17xy^2$

$g = 2x - 5y + 5x^2 + 5xy + 5y^2 - 6x^2y - 6xy^2$

How many common zeroes?

Bézout: ≤ 9. In general, $F = \sum_{\substack{i+j\leq d_1 \\ i,j\geq 0}} a_{ij}x^iy^j$, $G = \sum_{\substack{i+j\leq d_2 \\ i,j\geq 0}} b_{ij}x^iy^j \in \mathbb{C}[x,y]$

$\Rightarrow$ if $\{F=0\} \cap \{G=0\}$ is finite, then $|\{F=0\}\cap\{G=0\}| \leq d_1 \cdot d_2$,

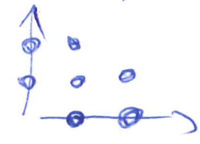
and for general a_{ij}, b_{ij} (i.e. open subset $\subseteq \mathbb{C}^N$, $N = \binom{d_1+2}{2} + \binom{d_2+2}{2}$)

$|\{F=G=0\}| = d_1 \cdot d_2$.

Kushnirenko: ≤ 6 on $(\mathbb{C}^*)^2$ (≤ 7 on $\mathbb{C}^2$). $\operatorname{supp} F := \{(i,j) \mid a_{ij} \neq 0\}$, then for F, G, s.t. $\operatorname{supp} F, G \subseteq \mathcal{A} \subseteq \mathbb{N}_0^2$

if $|\{F=G=0\}| < \infty$ then $|\{F=G=0\} \cap (\mathbb{C}^*)^2| \leq 2 \cdot \operatorname{Volume}(\operatorname{Conv}(\mathcal{A}))$

and for general a_{ij}, b_{ij} s.t. $\operatorname{supp} F, G \subseteq \mathcal{A}$ (open subset $\subseteq \mathbb{C}^{|\mathcal{A}|}$) $|\{F=G=0\}| = 2 \cdot \operatorname{Volume}(\operatorname{Conv}(\mathcal{A}))$

$\operatorname{supp} f = \operatorname{supp} g =$ , Volume $= 3$.

<u>Rk</u>: Bézout: $\underset{CH^1(\mathbb{P}^2)}{[F=0]} \cdot \underset{CH^1(\mathbb{P}^2)}{[G=0]} \in CH^2(\mathbb{P}^2) \overset{\deg}{\cong} \mathbb{Z}$, $\quad \mapsto d_1 \cdot d_2$

Kushnirenko: $CH^*(X)$ for some toric X.

Mostly follow "Toric varieties" Cox, Little, Schenck '11

Other references: "Introduction to toric varieties" Fulton '93

"The geometry of toric varieties" Danilov '78

Everything over $\mathbb{C}$.

II Affine toric varieties.

Def Affine varieties: $\mathrm{AffVar}_{\mathbb{C}} := \{\text{fin. gen. reduced (commutative) } \mathbb{C}\text{-alg}\}$
no nontrivial nilpotents

A - fin. gen. reduced $\mathbb{C}$-alg $\rightsquigarrow$ $\mathrm{Specm} := \{I \trianglelefteq A \mid I\text{-maximal}\} \overset{1:1}{\leftrightarrow} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(A, \mathbb{C})$
set of (rational) pts of the variety

(*) > $f \in A$, $I \in \mathrm{Specm}\, A$ $\rightsquigarrow$ value of f at I: $A \to A/_I \simeq \mathbb{C}$

~~...~~ If $f \in A$ and $f(I) = 0$ $\forall I \in \mathrm{Specm}\, A$ $\Rightarrow$ $f = 0$

$\rightsquigarrow$ A - functions ($\mathbb{C}$-valued) on $\mathrm{Specm}\, A$.

Zariski topology: $J \trianglelefteq A \rightsquigarrow Z(J) := \{I \in \mathrm{Specm} A \mid J \leq I\}$ - closed subsets

$f \in A \Rightarrow f$ - continuous in Zariski topology

(**) $J \trianglelefteq A \rightsquigarrow Z(J)$ carries the structure of an affine variety $\rightsquigarrow A/\sqrt{J}$, $\sqrt{J} = \{f \in A \mid f^n \in J \text{ for some } n\}$

$\varphi : A \to B \rightsquigarrow \mathrm{Specm}\, B \to \mathrm{Specm}\, A$ - continuous in Zariski top.
$I \mapsto \varphi^{-1}(I)$

If $\varphi_1, \varphi_2 : A \to B$ induce the same $\mathrm{Specm}\, B \to \mathrm{Specm}\, A$ $\Rightarrow$ $\varphi_1 = \varphi_2$

$\rightsquigarrow$ Affine varieties = "top. spaces + ~~particular~~ regular functions",
morphisms - continuous maps inducing homo-sms on regular functions.

Sometimes we will specify morphism only on points, ~~as it is straigh~~

Notation: $A \rightsquigarrow \mathrm{Spec}\, A$ - aff. variety
V - aff. variety $\rightsquigarrow \mathbb{C}[V]$ - regular functions.
$\mathbb{A}^n := \mathrm{Spec}\, \mathbb{C}[x_1, \ldots x_n]$

(*) Ex: $\mathrm{Specm}\, \mathbb{C}[x_1, \ldots x_n] \simeq \mathbb{C}^n$ - Nullstellensatz
$\langle x_1 - a_1, \ldots x_n - a_n \rangle \leftarrow (a_1, \ldots a_n)$

(Every aff. variety $\simeq$ closed subvariety of $\mathbb{A}^n$: $A \simeq \mathbb{C}[x_1, \ldots x_n]/_I$ — depends on the choice of generators

i.e. $V_1 = V_2 \setminus Z(J)$ for $J \trianglelefteq \mathbb{C}[V_2]$

(**) A morphism $V_1 \hookrightarrow V_2$ - open embedding, if it is an open embedd. of top. spaces, and $\mathbb{C}[V_1] \simeq \mathbb{C}[V_2][1/f \mid f \in \mathbb{C}[V_2] \setminus \bigcup_{I \in V_1} I]$
V_2 - irreducible

Ex: $\mathrm{Spec}\, \mathbb{C}[x, x^{-1}] \hookrightarrow \mathrm{Spec}\, \mathbb{C}[x] = \mathbb{A}^1$
$\mathbb{C}[x, x^{-1}] \hookleftarrow \mathbb{C}[x]$

V is irreducible if $V = Z_1 \cup Z_2$, $Z_1, Z_2 \leq V$ - closed $\Rightarrow$ $V = Z_1$ or $V = Z_2$.
$\Leftrightarrow$ $\mathbb{C}[V]$ is an integral domain.

~~Fact $V_1 \hookrightarrow V_2$ is an open embedding & V_2 is irreducible ... injective~~
~~Moreover $\mathbb{C}[V_1] = \mathbb{C}[V_2][1/f \mid f \notin \bigcup_{I \in V_1} I]$~~

Def $G_m := \mathrm{Spec}\, \mathbb{C}[x, x^{-1}]$, $G_m \times G_m \to G_m$ equips G_m with the structure of an alg. group (group object in $\mathrm{AffVar}_{\mathbb{C}}$)
$(t_1, t_2) \mapsto t_1 t_2$
$\mathbb{C}[x_1, x_2, x_1^{-1}, x_2^{-1}] \leftarrow \mathbb{C}[x, x^{-1}]$
$x_1 x_2 \leftarrow x$

An algebraic group T is a <u>torus</u> if $T \simeq \underbrace{G_m \times \ldots \times G_m}_{n}$ for some $n \in \mathbb{N}_0$

Def. An affine toric variety is an irreducible affine variety V with an open dense embedding $T \hookrightarrow V$ of a torus T s.t. the standard action of T on T extends to V.

Rk If the action extends, then it extends uniquely:

$$T\times V \to V \quad \longleftrightarrow \quad \mathbb{C}[V] \to \mathbb{C}[T]\otimes\mathbb{C}[V]$$

$$\begin{tikzcd} T\times T \arrow[r] \arrow[d] & T \arrow[d] \\ T\times V \arrow[r] & V \end{tikzcd} \quad \longleftrightarrow \quad \begin{tikzcd} \mathbb{C}[V] \arrow[r, dashed] \arrow[d, hook] & \mathbb{C}[T]\otimes\mathbb{C}[V] \arrow[d, hook] \\ \mathbb{C}[T] \arrow[r, hook] & \mathbb{C}[T]\otimes\mathbb{C}[T] \end{tikzcd}$$

Rk If $V \in \mathrm{AffVar}_{\mathbb{C}}$ admits the structure of a toric variety (i.e. $T \hookrightarrow V$ s.t. action extends) then it is essentially unique, i.e. for $T_1, T_2 \hookrightarrow V$ $\exists$ automorphism $V \xrightarrow{\varphi} V$ restricting to an iso-sm $\varphi|_{T_1}: T_1 \xrightarrow{\sim} T_2$ (Berchtold 2002)

How to construct affine toric varieties?

Def. • Lattice is an abelian group $\simeq \mathbb{Z}^n$ for some $n \in \mathbb{N}_0$

- Affine monoid is a fin. gen. monoid $\simeq$ to a submonoid of a lattice

S - aff. monoid $\Rightarrow \mathbb{Z}S := K(S) := \bigoplus_{s\in S} \mathbb{Z}\cdot[s] \big/ [s_1]+[s_2]=[s_1+s_2]$ - group completion.
$S \hookrightarrow \mathbb{Z}S \qquad \simeq S\times S/(s_1,s_2)\sim(s_1+s, s_2+s)$

• S - abelian monoid $\rightsquigarrow \mathbb{C}[S] := \{\sum_{s\in S} \overset{\in\mathbb{C}}{a_s} \overset{\text{symbol}}{\chi^s} \mid$ all but fin. many $a_s = 0\}$, $\chi^{s_1}\cdot\chi^{s_2} = \chi^{s_1+s_2}$

If S is fin.gen. then $\mathbb{C}[S]$ is fin.gen.

Ex: • $\mathbb{C}[\mathbb{N}_0^n] \simeq \mathbb{C}[x_1,\dots,x_n] \simeq \mathbb{C}[\mathbb{A}^n]$, $\chi^{e_i} \to x_i$
• $\mathbb{C}[\mathbb{Z}^n] \simeq \mathbb{C}[x_1^{\pm},\dots,x_n^{\pm}] \simeq \mathbb{C}[G_m^n]$, $\chi^{e_i} \to x_i$, $\chi^{-e_i} \to x_i^{-1}$
• $\mathbb{C}[\{0,2,3,4,\dots\}] = \mathbb{C} \oplus \bigoplus_{i\geq 2} \mathbb{C}\cdot\chi^i \simeq \mathbb{C}[u,v]/_{u^2-v^3}$

Lm $\mathrm{Hom}_{alg.gr.}(G_m^n, G_m^k) \simeq \mathbb{Z}^{n\cdot k}$

$$(t_1,\dots,t_n) \mapsto \left(\prod_{i=1}^n t_i^{m_{i1}}, \dots, \prod_{i=1}^n t_i^{m_{ik}}\right) \leftarrow \{m_{ij}\}$$

Pf: $\mathrm{Hom}_{alg.gr.}(G_m^n, G_m^k) \subseteq \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1^{\pm},\dots,x_k^{\pm}], \mathbb{C}[y_1^{\pm},\dots,y_n^{\pm}]) =$
$= (\mathbb{C}[y_1^{\pm},\dots,y_n^{\pm}]^*)^k = (\{\lambda\cdot y_1^{m_1}\dots y_n^{m_n} \mid \lambda\in\mathbb{C}^*, m_i \in \mathbb{Z}\}^*)^k$

$\lambda = 1$ since the homo-sm respects the neutral element of G_m^n

Cor The category of tori is dual to the cat. of lattices

$\{\text{tori}\} \rightleftarrows \{\text{lattices}\}^{op}$
$T \longmapsto \mathrm{Hom}(T, G_m) =: M(T)$ – "lattice of characters"
$\underset{T(M)}{\mathrm{Spec}(\mathbb{C}[M])} \longleftarrow M$

$\mathrm{Spec}\,\mathbb{C}[M] \times \mathrm{Spec}\,\mathbb{C}[M] \to \mathrm{Spec}\,\mathbb{C}[M]$
$\mathbb{C}[M]\otimes\mathbb{C}[M] \leftarrow \mathbb{C}[M]$
$\chi^s\otimes\chi^s \longleftarrow \chi^s$

Construction. S - affine monoid $\Rightarrow \mathbb{C}[S]$ is a fin. gen. integral domain and $\operatorname{Spec}\mathbb{C}[S]$ admits a canonical structure of an affine toric variety with $T = \mathbb{T}(\mathbb{Z}S)$.

Pf. $\mathbb{C}[S]$ is fin. generated by $x^{s_1}, \dots x^{s_r}$, where $s_1, \dots s_r$ - generators of S

$$\underset{\nearrow}{\mathbb{C}[S] \hookrightarrow} \mathbb{C}[\mathbb{Z}S] \simeq \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] \Rightarrow \text{integral domain}$$

induces an open embedding $\operatorname{Spec}\mathbb{C}[\mathbb{Z}S] \hookrightarrow \operatorname{Spec}\mathbb{C}[S]$.

The action extends:

$$\begin{tikzcd}
x^s & \mathbb{C}[S] \arrow[r, dashed] \arrow[d, hook] & \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[S] \arrow[d, hook] & x^s \otimes x^s \\
& \mathbb{C}[\mathbb{Z}S] \arrow[r] & \mathbb{C}[\mathbb{Z}S] \otimes \mathbb{C}[\mathbb{Z}S] & \\
& x^s \arrow[r, mapsto] & x^s \otimes x^s &
\end{tikzcd}$$

Prop. V - affine toric variety with $T \hookrightarrow V \Rightarrow V \underset{\text{as toric varieties}}{\simeq} \operatorname{Spec}\mathbb{C}[S]$ for some $S \subseteq M(T)$

Pf. $T \hookrightarrow V \Rightarrow$

$$\begin{tikzcd}
v \in \mathbb{C}[V] \arrow[r, hook] \arrow[d] & \mathbb{C}[T] \simeq \mathbb{C}[M(T)] \arrow[d] & \sum\limits_{a_i \in \mathbb{C}} a_i x^{\lambda_i} \\
\mathbb{C}[V] \otimes \mathbb{C}[M(T)] \arrow[r, hook] & \mathbb{C}[M(T)] \otimes \mathbb{C}[M(T)] & \\
\sum v_i \otimes x^{\lambda_i},\ v_i \in \mathbb{C}[V] \arrow[r] & \sum a_i \cdot x^{\lambda_i} \otimes x^{\lambda_i} &
\end{tikzcd}$$

$\Rightarrow v_i = a_i \cdot x^{\lambda_i} \in \mathbb{C}[V] \subseteq \mathbb{C}[M(T)]$

$S := \{\lambda \in M(T) \mid x^\lambda \in \mathbb{C}[V]\}$

$\Rightarrow \mathbb{C}[V] = \{\sum\limits_{\lambda \in S} a_\lambda x^\lambda \mid a_\lambda \in \mathbb{C}, \text{ allmost all } a_\lambda = 0\}$

S is clearly a submonoid of $M(T)$

$\mathbb{C}[V]$ is fin. gen. by $v_1 = \sum\limits_{\lambda \in \mathcal{A}_1} a_{1\lambda} x^\lambda, \dots, v_r = \sum\limits_{\lambda \in \mathcal{A}_r} a_{r\lambda} x^\lambda \Rightarrow$

$\Rightarrow S$ is generated by $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_r$ $\square$

Rk. V - aff. toric variety $\rightsquigarrow (\mathbb{C}^\times)^n \underset{\{\text{pts. of } T\}}{\cancel{}\ \operatorname{Spec}} T$ acts on the space of pts of $V \rightsquigarrow$ a representation of $(\mathbb{C}^\times)^n$ on $\mathbb{C}[V]$. $\mathbb{C}[V] \simeq \bigoplus\limits_{s \in S} \mathbb{C} \cdot x^s$ - eigenspace decomp.

Prop. S - affine monoid, $s_1, \dots s_r$ - generators $\rightsquigarrow$

$$\begin{tikzcd}
\mathbb{N}_0^r \arrow[r, two heads] \arrow[d, hook] & S \arrow[d, hook] \\
L \arrow[r, hook] & \mathbb{Z}^r \arrow[r, two heads, "\varphi"] & \mathbb{Z}S
\end{tikzcd}$$

$e_i \to s_i$, $L = \ker\varphi$

& $\mathbb{C}[\mathbb{N}_0^r] \overset{\Phi}{\twoheadrightarrow} \mathbb{C}[S]$

$\mathbb{C}[x_1, \dots x_r] \to$, $x_i \mapsto x^{s_i}$

Then $\ker\Phi = \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}_0^r, \alpha - \beta \in L \rangle$.

Pf. $I_L := \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}_0^r, \alpha - \beta \in L \rangle$

$\Phi(x^\alpha - x^\beta) = x^{\varphi(\alpha)} - x^{\varphi(\beta)} = 0$ since $\varphi(\alpha) = \varphi(\beta) \Rightarrow I_L \subseteq \ker\Phi$

Let ">" be lexigraphic order on $\mathbb{N}_0^r$, i.e. $(l_1, \dots l_r) > (l_1', \dots l_r')$ if $l_1 = l_1', \dots l_i = l_i', l_{i+1} > l_{i+1}'$

Pick $f \in \ker \varphi \setminus I_L$ with minimal leading monomial $= x^\alpha$,

$$f = c_\alpha x^\alpha + \sum_{\beta < \alpha} c_\beta \cdot x^\beta$$

$\varphi(x^\alpha) = x^{\sum \alpha_i s_i}$ where $\alpha = (\alpha_1, \dots, \alpha_r)$. $\varphi(f) = 0 \Rightarrow$

$\Rightarrow \exists \beta < \alpha$ s.t. $c_\beta \neq 0$ & $\sum \beta_i \cdot s_i = \sum \alpha_i s_i \Rightarrow \alpha - \beta \in L$

$\Rightarrow f - c_\alpha \cdot (x^\alpha - x^\beta) \in \ker \varphi \setminus I_L$ with smaller leading monomial $\square$

Def. Let $L \leq \mathbb{Z}^r$ be a sublattice.

$I_L := \langle x^\alpha - x^\beta \mid \alpha, \beta \in \mathbb{N}_0^S, \alpha - \beta \in L \rangle \leq \mathbb{C}[x_1, \dots x_r]$ - lattice ideal

Rk: $I_L = \langle x^{l_+} - x^{l_-} \mid l \in L \rangle$, where $(l_+)_i = \begin{cases} l_i, & l_i \geq 0 \\ 0, & l_i < 0 \end{cases}$, $(l_-)_i = \begin{cases} -l_i, & l_i < 0 \\ 0, & l_i \geq 0 \end{cases}$

Prop. $I_L \leq \mathbb{C}[x_1, \dots x_r]$ - lattice ideal $\Rightarrow \mathbb{C}[x_1, \dots x_r]/I_L \simeq \mathbb{C}[S]$, where $S := \mathbb{N}_0^r/\sim$, $\alpha \sim \beta \Leftrightarrow \alpha - \beta \in L$.

Pf: the same as above

Def. ~~[illegible]~~. I_L is toric if I_L is prime.

Rk (Exercise) I_L - toric $\Leftrightarrow$ S - affine monoid.

Prop. $I \trianglelefteq \mathbb{C}[x_1, \dots x_r]$ - toric $\Leftrightarrow$ I is prime and generated by binomials $\{x^\alpha - x^\beta\}$

Pf: "$\Rightarrow$" trivial

"$\Leftarrow$" $L := \{\alpha - \beta \mid x^\alpha - x^\beta \in I\}$ • ~~[illegible]~~. $x^\alpha - x^\beta \in I$, $x^{\alpha'} - x^{\beta'} \in I$.

$\forall \gamma \in \mathbb{N}_0^r$ s.t. $\alpha' - \beta + \gamma \in \mathbb{N}_0^r \rightsquigarrow x^{\alpha' - \beta + \gamma}(x^\alpha - x^\beta) + x^\gamma(x^{\alpha'} - x^{\beta'}) = x^{\alpha + \alpha' - \beta + \gamma} - x^{\beta' + \gamma} \in I$

$\Rightarrow x^\gamma (x^{\alpha + \alpha'} - x^{\beta + \beta'}) \in I \Rightarrow x^{\alpha + \alpha'} - x^{\beta + \beta'} \in I$

$x^\gamma \notin I$ because $\forall f \in I$ $f(1, \dots 1) = 0$

• $x^\alpha - x^\beta \in I \Rightarrow x^\beta - x^\alpha \in I \Rightarrow \beta - \alpha \in L$

$\Rightarrow L$ - lattice

$I \leq I_L$ - clear. $\alpha - \beta \in L \Rightarrow$ ~~[illegible]~~ $x^{\alpha + \gamma_1 - \gamma_2} - x^{\beta + \gamma_1 - \gamma_2} \in I$ for some $\gamma_1, \gamma_2 \in \mathbb{N}_0^r$

$\Rightarrow x^{\alpha + \gamma_1} - x^{\beta + \gamma_1} \in I \overset{I\text{-prime}}{\Rightarrow} x^\alpha - x^\beta \in I \Rightarrow I_L \leq I$ $\square$

Exercise: $\mathcal{A} \subseteq M(T)$, $\mathcal{A} = \{a_1, \dots a_r\}$, T-torus, $\varphi_\mathcal{A}: T \to \mathbb{G}_m \times \dots \times \mathbb{G}_m \hookrightarrow \mathbb{A}^{|\mathcal{A}|}$, $t \mapsto (a_1(t), \dots a_{|\mathcal{A}|}(t))$. $\Rightarrow Y_\mathcal{A} := \overline{\varphi_\mathcal{A}(T)}$ - aff. toric variety, $\mathbb{C}[Y_\mathcal{A}] \simeq \mathbb{C}[\mathbb{N}_0 \mathcal{A}]$

$\{$aff. toric varieties$\}$ $\rightleftarrows$ $\{$affine monoids$\}$ $\leftarrow$ / $\dashrightarrow$ (+ choice of generators) $\{$toric ideals$\}$

Rk: ~~[illegible]~~ S_1, S_2

check functoriality: $T_1 \hookrightarrow V_1$, $T_2 \hookrightarrow V_2$, $\varphi: V_1 \to V_2$ s.t. $\varphi|_{T_1}: T_1 \to T_2$ hom-sm; homo-sms $S_2 \to S_1$

Ex:

$\mathbb{A}^r$ — $\mathbb{N}_0^r$

$\mathbb{G}_m^r$ — $\mathbb{Z}^r$ — 0 — $\langle x_i y_i - 1 \rangle$ — $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{2n} \to \mathbb{Z}^n$, $e_i \to e_i + e_i'$; $e_i \to e_i$, $e_i' \to -e_i$

$\{0, 2, 3, \dots\} \leq \mathbb{Z}$ — $\langle x^3 - y^2 \rangle$ — $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$, $1 \to 3e_1 - 2e_2$; $e_1 \to 2$, $e_2 \to 3$

Def. A-integral domain, F-fraction field of A. $x\in F$ is <u>integral</u> over A if $\exists\, a_1,\dots,a_n\in A$ s.t. $x^n+a_1x^{n-1}+\dots+a_n=0$. A is <u>integrally closed</u> if $A=\{x\in F \mid x \text{ is int. over } A\}$.
An irreducible affine variety V is <u>normal</u> if $\mathbb{C}[V]$ is integrally closed.

Rk. A is int. closed $\Leftrightarrow$ $\forall\, I\in \operatorname{Specm} A$ $\ A_I:=\{f/g\in F \mid g\notin I\}$ is int. closed.

Ex: A-UFD $\Rightarrow$ A is int. closed $\Rightarrow$ $\mathbb{A}^{n_1}\times \mathbb{G}_m^{n_2}$ is normal

$V=\operatorname{Spec}\mathbb{C}[x,y]/(x^3-y^2)$ is not normal: $\bar y/\bar x\in\mathbb{C}(V)$, $(\bar y/\bar x)^2-\bar x=0$ but $\bar y/\bar x\notin\mathbb{C}[V]$.

Rk. V-normal $\Leftrightarrow$ V admits no non-trivial finite birational covers.

Rk (Hartogs theorem) V-normal variety, $W\subseteq V$ closed, $\dim W\le \dim V-2$, f-regular function on $V-W$ $\Rightarrow$ f extends to a regular function on V.

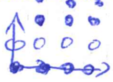
Fact V-irreducible affine variety, $A':=\{f\in\mathbb{C}(V)\mid f \text{ is integral over } A\}$ $\rightsquigarrow$
A' is a fin. gen. red. $\mathbb{C}$-alg & A' is int. closed.

Def V, A' as above $\rightsquigarrow$ $\operatorname{Spec}A'\to V$ - <u>normalization</u> of V

Ex: $\operatorname{Spec}\mathbb{C}[u]\to\operatorname{Spec}\mathbb{C}[x,y]/(x^3-y^2)$ - normalization.
$u^2 \mapsfrom x$
$u^3\mapsfrom y$

Def S-affine monoid. S is <u>saturated</u> (in $\mathbb{Z}S$) if $ks\in S$, $k\in\mathbb{N}_{>0}$, $s\in\mathbb{Z}S\Rightarrow s\in S$

Ex: $\mathbb{N}_0^n$ - saturated, $\{0,2,3,4,\dots\}$ - not saturated, $\langle e_1, e_1+2e_2, e_1+3e_2\rangle\subseteq\mathbb{Z}^2$ not saturated
$\langle e_1, e_1+2e_2\rangle$ - saturated



Thm. S-affine monoid. $\mathbb{C}[S]$ is int. closed $\Leftrightarrow$ S is saturated.

Pf: "$\Rightarrow$" $\mathbb{C}[S]\hookrightarrow\mathbb{C}[\mathbb{Z}S]\hookrightarrow F$ - fraction field
$ks\in S$, $k\in\mathbb{N}_{>0}$, $s\in\mathbb{Z}S$ $\rightsquigarrow$ $x^s\in F$ satisfies $(x^s)^k-\underset{\mathbb{C}[S]}{x^{ks}}=0$, i.e. x^s is integral over $\mathbb{C}[S]$ $\Rightarrow$ $x^s\in\mathbb{C}[S]$ $\Rightarrow s\in S$.
"$\Leftarrow$" later

Lm S-affine monoid, $M:=\mathbb{Z}S$, $M_{\mathbb{R}}:=M\otimes_{\mathbb{Z}}\mathbb{R}$. S-saturated $\Leftrightarrow$ $S=M\cap\operatorname{Cone}(S)$, where $\operatorname{Cone}(S)=\{\sum_{s\in S}a_s\cdot s\mid a_s\in\mathbb{R}_{\ge0}\}$

Pf: <u>Claim</u> (Exercise): $\mathcal{A}\subseteq M$, $|\mathcal{A}|<\infty$ $\Rightarrow$ $\operatorname{Cone}(\mathcal{A})\cap M=\{\sum_{s\in\mathcal{A}}a_s\cdot s\mid a_s\in\mathbb{Q}_{\ge0}\}\cap M$.
Hint: use Caratheodory theorem, allowing to assume that $\mathcal{A}$ is lin. independent.
"$\Leftarrow$" clear
"$\Rightarrow$" $S\subseteq M\cap\operatorname{Cone}(S)$ - clear. $m\in M\cap\operatorname{Cone}(S)$ $\rightsquigarrow$ $m=\sum_{s\in\mathcal{A}}a_s\cdot s$, $a_s\in\mathbb{Q}_{\ge0}$ $\rightsquigarrow$ $\exists N\in\mathbb{N}_{>0}$ s.t. $N\cdot a_s\in\mathbb{N}_{\ge0}$ $\forall s$
$\mathcal{A}$-gen. set for S.
$\Rightarrow N\cdot m\in S\Rightarrow m\in S$.

Def. M-lattice, $M_{\mathbb{R}}:=M\otimes_{\mathbb{Z}}\mathbb{R}$, $\sigma\subseteq M_{\mathbb{R}}$ is a (convex rational polyhedral) <u>cone</u> if $\sigma=\{\sum_{s\in\mathcal{A}}a_s\cdot s\mid a_s\in\mathbb{R}_{\ge0}\}$ for some finite $\mathcal{A}\subseteq M$.

<u>Gordan's lemma</u>: M-lattice, $\sigma\subseteq M_{\mathbb{R}}$ - cone $\Rightarrow$ $\sigma\cap M$ is an affine monoid.

Pf. $\sigma\cap M$ is a submonoid of M, so we need to show fin. gen.
Let $\mathcal{A}\subseteq M$, $|\mathcal{A}|<\infty$ be a gen. set for σ. $K:=\{\sum_{s\in\mathcal{A}}a_s\cdot s\mid 0\le a_s<1\}$ - bounded region in $M_{\mathbb{R}}\cong\mathbb{R}^n$
$\Rightarrow |K\cap M|<\infty$ since M is a lattice & $K\cap M\subseteq\sigma\cap M$. We claim that $K\cap M$ generates $\sigma\cap M$.
$\forall w\in\sigma\cap M\Rightarrow w=\sum_{s\in\mathcal{A}}a_s\cdot s\underset{a_s=[a_s]+\{a_s\}}{=}\sum_{s\in\mathcal{A}}[a_s]\cdot s+\sum_{s\in\mathcal{A}}\{a_s\}\cdot s$. $w,\sum[a_s]\cdot s\in M\Rightarrow\sum\{a_s\}\cdot s\in M$
$\Rightarrow\sum\{a_s\}\cdot s\in K\cap M\Rightarrow\square$

$$\left\{\begin{matrix}\text{normal affine}\\ \text{toric varieties}\end{matrix}\right\} \xrightarrow[\dashleftarrow]{} \left\{\begin{matrix}\text{saturated aff.}\\ \text{monoids}\end{matrix}\right\} \rightleftarrows \{\text{cones} \subseteq M_{\mathbb{R}}\} \rightleftarrows \{\text{cones} \subseteq M_{\mathbb{R}}^{\vee}\}$$

$$V_2 \to V_1 \longleftrightarrow S_1 \subseteq S_2 \longleftrightarrow \sigma_1 \subseteq \sigma_2 \longleftrightarrow \sigma_2' \subseteq \sigma_1'$$

Def. N-lattice, $\sigma \subseteq N_{\mathbb{R}}$-cone, $M := \mathrm{Hom}(N, \mathbb{Z})$, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Hom}_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R})$.
For $s \in M_{\mathbb{R}}$, $u \in N_{\mathbb{R}}$ put $\langle s, u\rangle := s(u) \in \mathbb{R}$ for the standard pairing.
$\sigma^{\vee} := \{ s \in M_{\mathbb{R}} \mid \langle s, u\rangle \geq 0 \ \forall u \in \sigma\}$

Fact (Farkas' theorem) $\sigma^{\vee}$ is a cone (convex, rational, polyhedral) and $(\sigma^{\vee})^{\vee} = \sigma$

$$S_{\sigma} := \sigma^{\vee} \cap M$$

Def. $\sigma \subseteq N_{\mathbb{R}}$ as above. For $s \in \sigma^{\vee}$ $H_s := \{u \in N_{\mathbb{R}} \mid \langle s, u\rangle = 0\}$ - supporting hyperplane of σ
$H_s^+ := \{u \in N_{\mathbb{R}} \mid \langle s, u\rangle \geq 0\}$

$$H_s \cap \sigma = \{u \in \sigma \mid \langle s, u\rangle = 0\} = \{\textstyle\sum \lambda_u \cdot u \mid \lambda_u \in \mathbb{R}_{\geq 0}, u \in \mathfrak{A} \text{ s.t. } \langle s, u\rangle = 0\}$$

gen. set for σ

- face of σ. $\Downarrow$ There is only fin. number of faces.

- every face is a cone

For $\tau, \sigma \subseteq N_{\mathbb{R}}$ cones write $\tau \preccurlyeq \sigma$ if τ is a face of σ.

- $\tau_1, \tau_2 \preccurlyeq \sigma \Rightarrow \tau_1 \cap \tau_2 \preccurlyeq \sigma$
- $\tau_1 \preccurlyeq \sigma$, $\tau_2 \preccurlyeq \tau_1 \Rightarrow \tau_2 \preccurlyeq \sigma$

Def $\sigma \subseteq N_{\mathbb{R}}$ as above. $\dim \sigma := \dim_{\mathbb{R}} \sigma + (-1)\cdot\sigma$.
For $\tau \preccurlyeq \sigma$ if $\dim \tau = \dim \sigma - 1$ then τ is a facet of σ.
$= 1$ edge

- $\tau \precneqq \sigma \Rightarrow \tau = \bigcap\limits_{\substack{\tau_i \text{- facet of } \sigma \\ \tau_i \supseteq \tau}} \tau_i$

- suppose $\sigma + (-1)\cdot\sigma = N_{\mathbb{R}} \Rightarrow \sigma = \bigcap\limits_{H_s \cap \sigma \text{- facet of } \sigma} H_s^+$

- $\tau \preccurlyeq \sigma \Rightarrow \tau^* := \{s \in \sigma^{\vee} \mid \langle s, u\rangle = 0 \ \forall u \in \tau\}$ is a face of $\sigma^{\vee}$; $\tau^{**} = \tau$
$\{\text{faces of } \sigma\} \xleftrightarrow{1:1} \{\text{faces of } \sigma^{\vee}\}$
$\tau_1 \preccurlyeq \tau_2 \Leftrightarrow \tau_2^* \preccurlyeq \tau_1^*$
$\dim \tau + \dim \tau^* = \dim_{\mathbb{R}} N_{\mathbb{R}}$

Def σ is strongly convex if $\{0\}$ is a face of σ $\left(\begin{matrix}\Leftrightarrow \sigma \cap (-1)\sigma = \{0\} \\ \Leftrightarrow \sigma^{\vee} + (-1)\cdot\sigma^{\vee} = M_{\mathbb{R}}\end{matrix}\right)$
σ is strongly convex $\Rightarrow$ edges are rays; $\rho \cap N$ has a minimal generator u_{ρ}.
$\{u_{\rho}\}_{\rho\text{-edge}}$ - minimal set of generators

Def σ - strongly convex cone. σ - smooth (or regular) is the set of min. generators extends to a $\mathbb{Z}$-basis of N.
σ - simplicial if the set of min. generators ~~extends~~ is lin. independent over $\mathbb{R}$.

Ex: $\langle e_1, e_2\rangle \subseteq \mathbb{R}^2$ - smooth
$\langle e_1 + 2e_2, 2e_2 + e_1\rangle$ - simpl., not smooth
$\langle e_1, e_2, e_3, e_1 + e_2 - e_3\rangle$ - not simpl & not smooth.

Lm σ - smooth $\Rightarrow \mathrm{Spec}\, \mathbb{C}[S_{\sigma}] \simeq \mathbb{A}^{r_1} \times \mathbb{G}_m^{r_2}$

PS $\sigma = \mathrm{cone}(e_1', \dots, e_{r_1}') \subseteq \mathbb{R}^{r_1 + r_2}$
$\rightsquigarrow \sigma^{\vee} = \mathrm{cone}(e_1, \dots, e_{r_1}, \pm e_{r_1+1}, \dots, \pm e_{r_1+r_2})$
$\Rightarrow S_{\sigma} = \mathbb{Z}^{r_1+r_2} \cap \sigma^{\vee} = \mathbb{N}_0^{r_1} \times \mathbb{Z}^{r_2}$

(*) **Def.** $T_1 \subseteq V_1$, $T_2 \subseteq V_2$ - aff. toric varieties. Morphism $\varphi : V_1 \to V_2$ - toric if $\varphi(T_1) \subseteq T_2$ and $\varphi|_{T_1} : T_1 \to T_2$ - homomorphism.

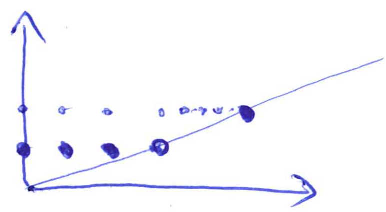
Exercise: $S_1 \to S_2$ - homo-sm of affine monoids $\Rightarrow \operatorname{Spec} \mathbb{C}[S_2] \to \operatorname{Spec} \mathbb{C}[S_1]$ is toric. Conversely, every toric morphism $\operatorname{Spec} \mathbb{C}[S_2] \to \operatorname{Spec} \mathbb{C}[S_1]$ is induced by a homo-sm $S_1 \to S_2$

Def $\sigma \subseteq N_{\mathbb{R}}$ - strongly convex cone, $\sigma + (-1)\sigma = N_{\mathbb{R}}$, $S_\sigma := \sigma^\vee \cap M$.

$\mathcal{H} := \{ m \in S_\sigma \mid m$ is irreducible, i.e. $m = m_1 + m_2$ in $S_\sigma \Rightarrow m_1 = 0$ or $m_2 = 0 \}$

Exercise:
- $\mathcal{H}$ - finite, generates S_σ
- $\mathcal{H}$ contains ray generators of $\sigma^\vee$
- $\mathcal{H}$ is a minimal gen-set of S_σ

– Hilbert basis of S_σ



Thm. S - affine monoid. $\mathbb{C}[S]$ is int. closed $\Leftrightarrow$ S is saturated.

Pf. "$\Rightarrow$" proved

"$\Leftarrow$" $S = \mathbb{Z}S \cap \operatorname{Cone}(S)$; $\operatorname{Cone}(S) = \bigcap\limits_{\substack{\tau\text{-facet} \\ \text{of Cone}(S)}} H_\tau^+ \Rightarrow S = \bigcap\limits_{\substack{\tau\text{-facet} \\ \text{of } \sigma}} S_\tau$, $S_\tau := \mathbb{Z}S \cap H_\tau^+$

$\Rightarrow \mathbb{C}[S] = \bigcap\limits_{\tau\text{-facet}} \mathbb{C}[S_\tau] \subseteq \mathbb{C}[\mathbb{Z}S]$

We have $S_\tau \simeq \mathbb{Z}^{n-1} \times \mathbb{N}_0$ (choose a basis of $M^\vee$ with $e_1 \sim H_\tau$)

$\Rightarrow \mathbb{C}[S_\tau] \simeq \mathbb{C}[x_1, x_2^\pm, \dots, x_n^\pm]$ - int. closed $\Rightarrow \mathbb{C}[S]$ - int. closed.

Rk. $V_\sigma := \operatorname{Spec} \mathbb{C}[\underset{\sigma^\vee \cap M}{S_\sigma}] \rightsquigarrow U_\sigma \leftarrow \coprod U_\rho$, where ρ - edge of σ ($\sim \rho^*$ - facet of $\sigma^\vee$)

$U_\rho \cong \operatorname{Spec} \mathbb{C}[x_1, x_2^\pm, \dots, x_n^\pm]$

Prop. S - affine monoid, $M_{\mathbb{R}} := \mathbb{Z}S \otimes_{\mathbb{Z}} \mathbb{R}$, $S' := \mathbb{Z}S \cap \operatorname{Cone}(S) \Rightarrow \operatorname{Spec} \mathbb{C}[S'] \to \operatorname{Spec} \mathbb{C}[S]$ - normalization

Pf: $\mathbb{C}[S] \hookrightarrow \mathbb{C}[S'] \hookrightarrow \mathbb{C}[\mathbb{Z}S] \subset F$ - fraction field of $\mathbb{C}[S]$.

$\mathbb{C}[S']$ - int. closed since S' is saturated.

$\mathbb{C}[S']$ is integral over $\mathbb{C}[S]$: pick $m \in S' \Rightarrow m \in \mathbb{Z}S \cap \operatorname{Cone}(S) \Rightarrow$

$\Rightarrow m \in \mathbb{Z}S \cap \{ \sum_{s \in S} a_s s \mid a_s \in \mathbb{Q}_{\geq 0} \} \Rightarrow \exists N$ s.t. $N \cdot m \in S \Rightarrow (x^m)^N - \underbrace{x^{Nm}}_{\in \mathbb{C}[S]} = 0$

$\Rightarrow x^m$ is integral over $\mathbb{C}[S]$ $\square$.

Ex: $S = \{0, 2, 3, \dots\} \rightsquigarrow \operatorname{Cone}(S) \cap \mathbb{Z}S = \{0, 1, 2, \dots\} \simeq \mathbb{N}_0 \Rightarrow \operatorname{Spec} \mathbb{C}[\mathbb{N}_0] \to \operatorname{Spec} \mathbb{C}[S]$ - normalization

$$\left\{\begin{matrix}\text{normal affine} \\ \text{toric varieties}\end{matrix}\right\} \underset{\mathbb{C}[-]}{\overset{}{\rightleftarrows}} \left\{\begin{matrix}\text{saturated} \\ \text{monoids}\end{matrix}\right\} \underset{M \cap (-)}{\overset{\operatorname{Cone}(-)}{\rightleftarrows}} \left\{\begin{matrix}\text{cones of} \\ \text{full dimension}\end{matrix}\right\} \overset{(-)^\vee}{\longleftrightarrow} \left\{\begin{matrix}\text{strongly} \\ \text{convex} \\ \text{cones}\end{matrix}\right\}$$

Prop. $\sigma \subseteq N_{\mathbb{R}}$ - strongly convex cone, $\tau \leq \sigma \rightsquigarrow \exists s \in \sigma^\vee \cap M$ s.t. $\tau = H_s \cap \sigma$

$\Rightarrow \mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma][x^{-s}] \subseteq \mathbb{C}[M]$

Pf. $\tau \leq \sigma \Rightarrow \tau^\vee \supseteq \sigma^\vee \Rightarrow S_\tau \supseteq S_\sigma$; $\langle s, u\rangle = 0\ \forall u \in \tau \Rightarrow -s \in \tau^\vee$
$\Rightarrow S_\sigma + \mathbb{Z}\cdot s \subseteq S_\tau$. Pick $m \in S_\tau$, $\mathcal{A} \subseteq \sigma$ - fin. set of generators of σ.
$\exists N \in \mathbb{N}_0$ s.t. $|\langle m, u\rangle| \leq N\ \forall u \in \mathcal{A} \Rightarrow \langle m + Ns, u\rangle \geq 0\ \forall u \in \mathcal{A}$
$\left(\begin{array}{l} \text{If } \langle s,u\rangle > 0 \Rightarrow \text{clear} \\ \text{If } \langle s,u\rangle = 0 \Rightarrow u \in \tau \Rightarrow \langle m,u\rangle \geq 0 \end{array}\right)$
$\Rightarrow m + N\cdot s \in \sigma^\vee \Rightarrow m + Ns \in S_\sigma \Rightarrow m \in S_\sigma + \mathbb{Z}\cdot s \Rightarrow S_\sigma + \mathbb{Z}s = S_\tau$ ☐

Rk. $\tau \leq \sigma \Rightarrow U_\tau \hookrightarrow U_\sigma$ - open embedding.
In particular, if $\sigma_1, \sigma_2 \subseteq N_\mathbb{R}$ and $\sigma_1 \cap \sigma_2 = \tau$, $\tau \leq \sigma_1$, $\tau \leq \sigma_2$
$\Rightarrow U_{\sigma_1} \hookleftarrow U_\tau \hookrightarrow U_{\sigma_2}$

Def. $V \in \mathrm{AffVar}_\mathbb{C}$, V-irreducible, $p \in V$ - point (i.e. $p \leq \mathbb{C}[V]$ - max ideal)
$\rightsquigarrow T_V|_p := \mathrm{Hom}_\mathbb{C}(p/p^2, \mathbb{C})$ - (Zariski) <u>tangent space</u> of V at p

Fact: $\mathcal{O}_{V,p} := \mathbb{C}[V]_p := \{ f/g \in \mathbb{C}(V) \mid g \notin p\}$ - <u>local</u> ring of V at p,
$m_{V,p} := \{ f/g \in \mathbb{C}(V) \mid f \in p, g \notin p\}$ - unique max. ideal in $\mathcal{O}_{V,p}$.
$\Rightarrow p/p^2 \simeq m_{V,p}/m_{V,p}^2$.

p - <u>smooth</u> pt of V if $\dim_\mathbb{C} T_V|_p = \dim V$ ($\Leftrightarrow \mathcal{O}_{V,p}$ - regular local ring)
V - <u>smooth</u> if all points are smooth.

Ex: $\mathrm{Spec}\, \mathbb{C}[x,y]/(x^3 - y^2)$ is not smooth at $(0,0)$.

Fact V-smooth $\Rightarrow$ V-normal.

Thm. $\sigma \leq N_\mathbb{R}$ - strongly convex cone. U_σ is smooth $\Leftrightarrow$ σ is smooth (i.e. has a gen. set that extends to a basis of N)

Pf "$\Leftarrow$" $U_\sigma \simeq \mathbb{A}^{r_1} \times \mathbb{G}_m^{r_2}$ - smooth.
"$\Rightarrow$" • Suppose $\dim \sigma = \dim N_\mathbb{R} \Rightarrow \sigma^\vee$ - strongly convex, $\mathcal{H} := \{ s \in S_\sigma \mid s\text{-irred.}\}$ - Hilbert basis of S_σ
$\rightsquigarrow I := \langle x^s \mid s \in \mathcal{H}\rangle = \{ \sum a_s x^s \mid s \in S_\sigma \setminus \{0\}, a_s \in \mathbb{C}\}$ - maximal ideal in $\mathbb{C}[S_\sigma]$
$I/I^2 \simeq \bigoplus_{s \in \mathcal{H}} \mathbb{C}\bar{x}^s$. $\mathbb{C}[S_\sigma]$ - smooth $\Rightarrow |\mathcal{H}| = \dim U_\sigma = \dim M_\mathbb{R}$
$\mathcal{H}$ gen. $S_\sigma \Rightarrow \mathbb{Z}\mathcal{H} = \mathbb{Z}S_\sigma = M \Rightarrow \mathcal{H}$ is a $\mathbb{Z}$-basis for M
$\Rightarrow \sigma^\vee$ is smooth $\Rightarrow \sigma$ is smooth.

• $W := \sigma + (-1)\sigma \leq N_\mathbb{R}$, $N_1 := W \cap N$. N_1 is saturated in N $\Rightarrow$
$\Rightarrow N/N_1 \simeq \mathbb{Z}^r \Rightarrow N \simeq N_1 \oplus N_2$ for some sublattice $N_2 \leq N$,
$\sigma \subseteq (N_1)_\mathbb{R}$, σ_{N_1} - σ viewed as a cone in N_1.
$M = N_1^\vee \oplus N_2^\vee$, $\sigma^\vee = \sigma_{N_1}^\vee \oplus (M_2)_\mathbb{R}$, $M_2 := N_2^\vee$
$\Rightarrow S_\sigma = S_{\sigma_{N_1}} \times \mathbb{Z}^r \Rightarrow U_\sigma \simeq U_{\sigma_{N_1}} \times \mathbb{G}_m^r$
U_σ - smooth $\underset{\text{Exercise}}{\Rightarrow}$ $U_{\sigma_{N_1}}$ - smooth $\Rightarrow$ σ_{N_1} - smooth.

III Normal toric varieties

Def. V-affine variety, $f \in \mathbb{C}[V] \rightsquigarrow V_f := \operatorname{Spec} \mathbb{C}[V][f^{-1}] \overset{\text{open}}{\hookrightarrow} V$ - <u>principal open subset</u>.
$\{V_f\}_{f \in \mathbb{C}[V]}$ - base of the Zariski topology on V

V-affine variety, $U \subseteq V$-open subset $\varphi : U \to \mathbb{C}$ - regular if $\forall p \in U$ $\exists f \in \mathbb{C}[V]$ s.t. $p \in V_f \subseteq U$ & $\varphi|_{V_f}$ - regular, i.e. $\varphi|_{V_f} \in \mathbb{C}[V_f]$.

$\rightsquigarrow$ <u>sheaf</u> $\mathcal{O}_V$ of regular functions on V: $U \to \mathcal{O}_V(U) := \{\varphi : U \to \mathbb{C} \mid \varphi\text{-regular}\}$

Rk $\mathcal{O}_V(V) = \mathbb{C}[V]$, $\mathcal{O}_V(V_f) = \mathbb{C}[V][f^{-1}]$

Def V_1, V_2 - aff. varieties, $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ - open subsets. A map $\varphi : U_1 \to U_2$ is a (<u>regular</u>) <u>morphism</u> if it defines a map $\mathcal{O}_{V_2}(U_2) \to \mathcal{O}_{V_1}(U_1)$

$$\operatorname{Maps}(U_2, \mathbb{C}) \xrightarrow{(-)\circ\varphi} \operatorname{Maps}(U_1, \mathbb{C})$$

Rk. • $U_1 = V_1$, $U_2 = V_2 \rightsquigarrow \operatorname{Mor}(V_1, V_2) = \operatorname{Hom}(\mathbb{C}[V_2], \mathbb{C}[V_1])$
• $\mathcal{O}_V(U) = \operatorname{Mor}(U, \mathbb{A}^1)$

$\varphi : U_1 \to U_2$ is an isomorphism if φ is bijective and $\varphi^{-1} : U_2 \to U_1$ - morphism

Def. $\{V_\alpha\}_{\alpha \in \mathcal{A}\text{-finite}}$ - aff. varieties, $\{U_{\beta\alpha} \subseteq V_\alpha\}_{\alpha, \beta \in \mathcal{A}}$ - open subsets, $\varphi_{\beta\alpha} : U_{\beta\alpha} \xrightarrow{\sim} U_{\alpha\beta}$
s.t. • $\varphi_{\alpha\beta} = \varphi_{\beta\alpha}^{-1}$ • $\varphi_{\beta\alpha}(U_{\beta\alpha} \cap U_{\gamma\alpha}) = V_{\alpha\beta} \cap V_{\gamma\beta}$ & $\varphi_{\gamma\alpha} = \varphi_{\gamma\beta} \circ \varphi_{\beta\alpha}$ on $U_{\beta\alpha} \cap U_{\gamma\alpha}$

$\rightsquigarrow X := (\coprod V_\alpha) / a \sim \varphi_{\beta\alpha}(a)$ $\forall \alpha, \beta, a \in V_{\beta\alpha}$. Topology descends from $\coprod V_\alpha$

$V_\alpha \to X$ - homeomorphism on the image, the image is open in X

$\rightsquigarrow V_\alpha \subseteq X$ - open subsets, $X = \bigcup V_\alpha$.

Structure sheaf: $U \subseteq X$-open $\rightsquigarrow \mathcal{O}_X(U) = \{f : U \to \mathbb{C} \mid f|_{U \cap V_\alpha}\text{-regular } \forall \alpha\}$

$(X, \mathcal{O}_X)$ - pre-variety.

Ex: 1) $\begin{array}{c} x \;\; \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \\ \downarrow \wr\!\downarrow \\ x^{-1} \;\; \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \end{array} \rightsquigarrow \mathbb{P}^1$ 2) $\begin{array}{c} x \;\; \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \\ \downarrow \wr\!\downarrow \\ x \;\; \mathbb{G}_m \hookrightarrow \mathbb{A}^1 \end{array} \rightsquigarrow$ line with a double point.

Def $X = \bigcup V_\alpha$, $Y = \bigcup V'_\beta$. A <u>morphism</u> $\varphi : X \to Y$ is a Zariski continuous map such that $\varphi|_{V_\alpha \cap \varphi^{-1}(V'_\beta)} : V_\alpha \cap \varphi^{-1}(V'_\beta) \to V'_\beta$ - morphism $\forall \alpha, \beta$.

Def. $U \subseteq X = \bigcup V_\alpha$ - open $\rightsquigarrow U \cap V_\alpha$ - open in $V_\alpha \rightsquigarrow U \cap V_\alpha = \bigcup_{f \in S\text{-finite}} (V_\alpha)_f$ $\Rightarrow U$ has a canonical structure of a pre-variety

$U \subseteq X = \bigcup V_\alpha$ - closed $\rightsquigarrow U \cap V_\alpha$ - closed in $V_\alpha \rightsquigarrow U \cap V_\alpha$ - affine variety $\Rightarrow$

Def. X - pre-variety. X is <u>separated</u> if $\Delta(X) \subseteq X \times X$ is closed. <u>Variety</u> is a <u>separated pre-variety</u>

<u>Facts</u>: • Pre variety is separated $\Leftrightarrow$ is Hausdorff in the strong topology
• X-variety, $f, g : Y \to X$ - morphisms $\Rightarrow \{y \in Y \mid f(y) = g(y)\}$ - closed in Y.
• X-variety, $U, V \subseteq X$ - affine open $\Rightarrow U \cap V$ - affine.

Def X-variety, $p\in X \rightsquigarrow \mathcal{O}_{X,p} := \varinjlim_{p\in U} \mathcal{O}_X(U) = \{f_U : U \to \mathbb{C}\text{-regular} \mid p\in U\subseteq X\}/\sim$ $\quad f_U\sim g_{U'}$ if $f_U|_{U\cap U'} = g_{U'}|_{U\cap U'}$

- local ring of X at p

$m_{X,p} := \{f\in \mathcal{O}_{X,p} \mid f(p)=0\}$ - maximal ideal

X-irreducible $\rightsquigarrow \mathbb{C}(X) = \varinjlim_{\emptyset\neq U\subseteq X} \mathcal{O}_X(U)$ - field of rational functions

Def. X-irred. variety. X-normal if $\mathcal{O}_{X,p}$-normal $\forall p\in X$.

Fact. $X=\bigcup V_\alpha$ - normal $\Leftrightarrow V_\alpha$ - normal $\forall \alpha$.

Pf: $\mathbb{C}[V_\alpha] = \bigcap_{p\in V_\alpha} \mathcal{O}_{X,p}$ □

Def. X-smooth if $\forall p\in X$ $\quad \underbrace{\dim_{\mathbb{C}} (m_{X,p}/m_{X,p}^2)^\vee}_{T_X|_p} = \dim X$.

Def A toric variety X is an irreducible variety X with a Zariski open dense subset $T\subseteq X$ with T being a torus, such that the standard action $T\times T\to T$ extends to $T\times X\to X$.

Rk If the action extends, then it does so uniquely: $\varphi_1,\varphi_2 : T\times X\to X$

$T\times X$ - separated $\Rightarrow \varphi_1=\varphi_2$ on a closed subset of $T\times X$

$\varphi_1=\varphi_2$ on $T\times T$ $\quad\Rightarrow$ □

Def. N-lattice, $N_{\mathbb{R}} := N\otimes_{\mathbb{Z}}\mathbb{R}$. A fan in $N_{\mathbb{R}}$ is $\Sigma=\{\sigma\subseteq N_{\mathbb{R}}$ - strongly convex cones$\}$ s.t.

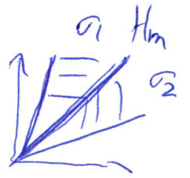
- Σ-finite • $\sigma\in\Sigma$, $\tau\preceq\sigma \Rightarrow \tau\in\Sigma$ • $\sigma_1,\sigma_2\in\Sigma \Rightarrow \sigma_1\cap\sigma_2 \preceq \sigma_1,\sigma_2$.

The support of Σ is $|\Sigma| = \bigcup_{\sigma\in\Sigma}\sigma \subseteq N_{\mathbb{R}}$

$\Sigma(r) := \{\sigma\in\Sigma \mid \dim\sigma = r\}$

~~Construction~~

Lm. $\sigma_1,\sigma_2\in\Sigma$, $\tau=\sigma_1\cap\sigma_2 \Rightarrow S_\tau = S_{\sigma_1}+S_{\sigma_2}$



Pf: $S_{\sigma_1}+S_{\sigma_2} = M\cap\sigma_1^\vee + M\cap\sigma_2^\vee \subseteq M\cap(\sigma_1^\vee+\sigma_2^\vee) = M\cap(\sigma_1\cap\sigma_2)^\vee = M\cap\tau^\vee = S_\tau$

Fact: $\exists\, m\in \sigma_1^\vee\cap(-\sigma_2)^\vee\cap M$ s.t. $H_m\cap\sigma_1 = H_m\cap\sigma_2 = \tau$. - "separation lemma"

$S_\tau = S_{\sigma_1} + \mathbb{Z}\cdot(-m) \subseteq S_{\sigma_1}+S_{\sigma_2}$ □

Construction Σ - fan in $N_{\mathbb{R}}$. $\rightsquigarrow \{U_\sigma\}_{\sigma\in\Sigma}$, $U_\sigma = \operatorname{Spec}\mathbb{C}[S_\sigma]$. $\sigma_1,\sigma_2\in\Sigma$

$\Rightarrow (U_{\sigma_1})_{\chi^m} \subseteq U_{\sigma_1}$

$\varphi_{\sigma_2\sigma_1}$: $(U_{\sigma_1})_{\chi^m} \supseteq U_\tau \subseteq (U_{\sigma_2})_{\chi^m} \subseteq U_{\sigma_2}$

Exercise: $\{\varphi_{\sigma_2\sigma_1}\}_{\sigma_2,\sigma_1\in\Sigma}$ satisfy compatibility conditions

$\Rightarrow X_\Sigma = \bigcup_{\sigma\in\Sigma} U_\sigma$ - variety; $U_{\{0\}} \cong \mathbb{G}_m^n \subseteq X_\Sigma$

Thm. X_Σ - normal toric variety.

Pf. $\tau\preceq\sigma \rightsquigarrow U_\tau \hookrightarrow U_\sigma$ - toric $\Rightarrow$ the actions $U_{\{0\}}\times U_\sigma\to U_\sigma$ glue into $U_{\{0\}}\times X_\Sigma\to X_\Sigma$

X_Σ - irreducible since all U_σ are irreducible

X_Σ - normal since all U_σ are normal.

X-separated: sufficient $\Delta: U_\tau\to U_{\sigma_1}\times U_{\sigma_2}$, $\tau=\sigma_1\cap\sigma_2$ - closed. comes from $\mathbb{C}[S_{\sigma_1}]\otimes\mathbb{C}[S_{\sigma_2}] \to \mathbb{C}[S_\tau] = \mathbb{C}[S_{\sigma_1}+S_{\sigma_2}]$ - surjective $\Rightarrow$ closed □

Proposition Σ - fan in $N_{\mathbb{R}}$. X_Σ - smooth $\Leftrightarrow$ all cones in Σ are smooth
(i.e. minimal generators form a subbasis in N)

Pf. $X_\Sigma = \bigcup_{\sigma\in\Sigma} U_\sigma$ - smooth $\Leftrightarrow$ U_σ - smooth $\forall \sigma\in\Sigma$
$\Leftrightarrow$ σ - smooth $\forall\sigma\in\Sigma$.

Examples:

- Dimension 1: possible cones $\underset{\tau}{\{0\}} \subseteq \mathbb{R}$, $\underset{\sigma_1}{[0,\infty)} \subseteq \mathbb{R}$, $\underset{\sigma_2}{(-\infty,0]} \subseteq \mathbb{R}$

fans: $\{\tau\}$, $\{\tau,\sigma_1\}$, $\{\tau,\sigma_2\}$, $\{\tau,\sigma_1,\sigma_2\}$.

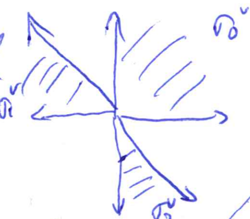
$S_\tau \simeq G_m$ $\quad S_{\sigma_1} \simeq \mathbb{A}^1$ $\quad S_{\sigma_2} \simeq \mathbb{A}^1$

$$S_{\sigma_1} \cup_{S_\tau} S_{\sigma_2} = \operatorname{Spec}\mathbb{C}[t] \underset{\operatorname{Spec}\mathbb{C}[t,t^{-1}]}{\cup} \operatorname{Spec}\mathbb{C}[t^{-1}] \simeq \mathbb{P}^1 \quad \operatorname{Spec}\mathbb{C}[\cancel{x_0}] \quad \operatorname{Spec}\mathbb{C}[\cancel{x_1}]$$

$\cancel{x_0} \to t \quad \operatorname{Spec}\mathbb{C}[t,t^{-1}] \quad \cancel{x_1}\ t^{-1}$

- $N=\mathbb{Z}^2$, $N_{\mathbb{R}} \simeq \mathbb{R}^2$ $\quad \Sigma = \{\sigma_0,\sigma_1,\sigma_2,\sigma_0\cap\sigma_1,\sigma_0\cap\sigma_2,\sigma_1\cap\sigma_2,\sigma_0\cap\sigma_1\cap\sigma_2\}$

dual $\rightsquigarrow$



$U_{\sigma_0} = \operatorname{Spec}\mathbb{C}[x_1,x_2]$ $\qquad \mathbb{C}[x_1,x_2][x_1^{-1}] \simeq \mathbb{C}[x_1^{-1},x_1^{-1}x_2][x_1]$

$U_{\sigma_1} = \operatorname{Spec}\mathbb{C}[x_1^{-1},x_1^{-1}x_2]$ $\qquad \mathbb{C}[x_1,x_2][x_2^{-1}] \simeq \mathbb{C}[x_2^{-1},x_1x_2^{-1}][x_2]$

$U_{\sigma_2} = \operatorname{Spec}\mathbb{C}[x_2^{-1},x_1x_2^{-1}]$ $\qquad \mathbb{C}[x_1^{-1},x_1^{-1}x_2][x_1x_2^{-1}] \simeq$
$\simeq \mathbb{C}[x_2^{-1},x_1x_2^{-1}][x_1^{-1}x_2]$

$\mathbb{A}^2 \sqcup \mathbb{A}^2 \sqcup \mathbb{A}^2$
$G_m\times\mathbb{A}^1 \quad G_m\times\mathbb{A}^1 \quad G_m\times\mathbb{A}^1$

$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2$

$[y_0:y_1:y_2]$ $\quad \left[\frac{y_0}{y_2}:\frac{y_1}{y_2}:1\right]$ $\quad \left[1:\frac{y_1}{y_0}:\frac{y_2}{y_0}\right]$ $\quad \left[\frac{y_0}{y_1}:1:\frac{y_2}{y_1}\right]$

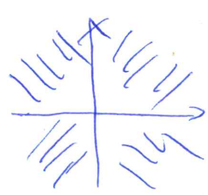
$$\boxed{\begin{aligned} x_1 &\to \frac{y_0}{y_2} \\ x_2 &\to \frac{y_1}{y_2}\end{aligned}}$$

$\mathbb{C}[v_1,v_2]$ $\qquad \mathbb{C}[w_1,w_2]$ $\qquad \Rightarrow X_\Sigma \simeq \mathbb{P}^2$

$\mathbb{C}[v_1,v_2,v_1^{-1}] \simeq \mathbb{C}[w_1,w_2,w_2^{-1}]$

$v_1 \longmapsto w_2^{-1}$
$v_2 \longmapsto w_1w_2^{-1}$

- $N=\mathbb{Z}^n$, $N_{\mathbb{R}}\simeq\mathbb{R}^n$, $\{e_1,\dots e_n\}$ - basis, $e_0 := -e_1-e_2-\dots-e_n$. $I\subseteq\{0,\dots,n\}$
$\rightsquigarrow \sigma_I := \operatorname{Cone}(\{e_i\}_{i\in I})$ $\quad \Sigma := \{\sigma_I\}_{I\subsetneq\{0,\dots,n\}}$ $\overset{\text{Exercise}}{\rightsquigarrow} X_\Sigma \simeq \mathbb{P}^n$

- 

$X_\Sigma \simeq \mathbb{P}^1\times\mathbb{P}^1$

$\operatorname{Spec}\mathbb{C}[x_1^{-1},x_2] \hookrightarrow \underset{\mathbb{P}^1\times\mathbb{A}^1}{} \hookleftarrow \operatorname{Spec}\mathbb{C}[x_1,x_2]$

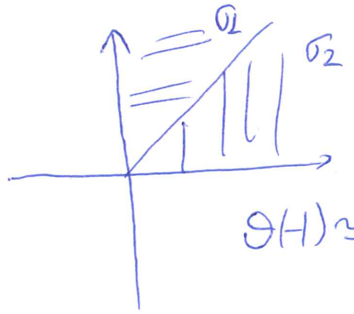
$\mathbb{P}^1\times\mathbb{P}^1$

$\operatorname{Spec}\mathbb{C}[x_1^{-1},x_2^{-1}] \hookrightarrow \underset{\mathbb{P}^1\times\mathbb{A}^1}{} \hookleftarrow \operatorname{Spec}\mathbb{C}[x_1,x_2^{-1}]$

Exercise: Σ_1 in $(N_1)_\mathbb{R}$, Σ_2 in $(N_2)_\mathbb{R}$ $\rightsquigarrow$ $\underset{\{\sigma_1\times\sigma_2 \mid \sigma_1\in\Sigma_1,\ \sigma_2\in\Sigma_2\}}{\underset{\shortparallel}{\Sigma_1\times\Sigma_2}}$ in $(N_1\times N_2)_\mathbb{R}$

$$\Longrightarrow X_{\Sigma_1\times\Sigma_2}\simeq X_{\Sigma_1}\times X_{\Sigma_2}.$$

•



$U_{\sigma_1}=\operatorname{Spec}\mathbb{C}[X_1, X_1^{-1}X_2]$

$U_{\sigma_2}=\operatorname{Spec}\mathbb{C}[X_2, X_1X_2^{-1}]$ ($\mathbb{A}^2$)

$\mathbb{C}[X_1,X_1^{-1}X_2][X_1X_2^{-1}]\simeq$
$\simeq\mathbb{C}[X_2, X_1X_2^{-1}][X_1^{-1}X_2]$

$$\mathcal{O}(-1)\simeq Bl_{(0,0)}\mathbb{A}^2 := \{(l,p)\mid p\in l\}\subseteq \mathbb{P}^1\times\mathbb{A}^2$$

$\downarrow$ $\mathbb{P}^1$

$$=\{([y_0:y_1],(x_0,x_1))\mid x_1\cdot y_0 = y_1\cdot x_0\}$$

$$\begin{tikzcd}
\mathbb{C}[v_0,w_0] & \mathbb{A}^1\times\mathbb{A}^1 \arrow[r, hook] \arrow[d, hook] & Bl_{(0,0)}\mathbb{A}^2 \arrow[d, hook] & \mathbb{A}^1\times\mathbb{A}^1 \arrow[l, hook'] \arrow[d, hook] & \mathbb{C}[v_1,w_1] \\
& \mathbb{A}^1\times\mathbb{A}^2 \arrow[r, hook] & \mathbb{P}^1\times\mathbb{A}^2 & \mathbb{A}^1\times\mathbb{A}^2 \arrow[l, hook'] &
\end{tikzcd}$$

$X_1^{-1}X_2 \mapsto v_0$, $X_1 \mapsto w_0$; $X_1X_2^{-1}\mapsto v_1$, $X_2 \mapsto w_1$

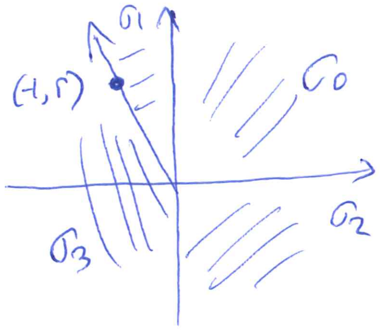
$\mathbb{C}[v_0,w_0] \to ([v_0:1],(v_0w_0, w_0))$

$\mathbb{C}[v_1,w_1] \to ([1:v_1],(w_1, v_1w_1))$

• Exercise: $N:=\mathbb{Z}^n$, $(e_1,\dots,e_n)$-basis, $e_0:=e_1+\dots+e_n$, $I\subseteq\{0,\dots,n\}\rightsquigarrow \sigma_I=\mathrm{Cone}(\{e_i\}_{i\in I})$

$$\Sigma:=\{\sigma_I\}_{\{1,\dots,n\}\not\subseteq I\subseteq\{0,\dots,n\}} \Longrightarrow X_\Sigma\simeq Bl_0\mathbb{A}^n := \{(l,p)\mid p\in l\}\subseteq\mathbb{P}^{n-1}\times\mathbb{A}^n$$

•



$\Sigma_r=\{\sigma_0,\sigma_1,\sigma_2,\sigma_3$ & faces$\}$.

$\Rightarrow X_{\Sigma_r}=:\mathcal{H}_r$ - Hirzebruch surfaces.

$$\begin{array}{ccc}
\operatorname{Spec}\mathbb{C}[X_1^{-1}, X_1^{r}X_2]\cup\operatorname{Spec}\mathbb{C}[X_1,X_2] & & \mathbb{P}^1 \\
\supset \qquad\qquad \subset & & | \\
\operatorname{Spec}\mathbb{C}[X_1^{-1}, X_1^{-r}X_2^{-1}]\cup\operatorname{Spec}\mathbb{C}[X_1,X_2^{-1}] & & \mathcal{H}_r\simeq\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(-r)) \\
\downarrow & & \downarrow \\
\operatorname{Spec}\mathbb{C}[X_1^{-1}]\cup\operatorname{Spec}\mathbb{C}[X_1] & & \mathbb{P}^1
\end{array}$$

Def. Σ in $N_\mathbb{R}$, Σ' in $N'_\mathbb{R}$ - fans. A morphism of fans $\Sigma\to\Sigma'$ is a homomorphism $\varphi: N\to N'$ s.t. $\forall\sigma\in\Sigma$ $\varphi_\mathbb{R}(\sigma)\subseteq\sigma'$ for some $\sigma'\in\Sigma'$.

~~Exercise~~ X, X' - toric varieties, $\varphi: X\to X'$ - toric morphism if $\varphi(T)\subseteq T'$ and $\varphi|_T: T\to T'$ hom.

Exercise: • $\varphi:\Sigma\to\Sigma'$ - morphism of fans $\Rightarrow$ $\Phi_\varphi: X_\Sigma=\bigcup U_\sigma\to\bigcup U_{\sigma'}=X_{\Sigma'}$ - toric morphism.

Later • $X_\Sigma\xrightarrow{\Phi}X_{\Sigma'}$ - toric morphism $\Rightarrow\exists\varphi:\Sigma\to\Sigma'$ s.t. $\Phi=\Phi_\varphi$.

Thm X-normal toric variety $\Rightarrow X \simeq X_\Sigma$ for a fan Σ.

Thm (Sumihiro) $T \le X$ - ~~tor~~ normal toric variety, $x \in X \Rightarrow \exists x \in U \le X$-open s.t. U-affine toric variety.

Pf (sketch):

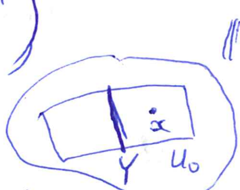
- pick $x \in W \subseteq X$ -affine open s.t. $X \setminus W = \bigcup_{i=1}^{m} Z_i$, $Z_i \le X$-closed, irreducible, of codimension 1

  $\rightsquigarrow U_0 := \bigcup_{t \in T} tW \subseteq X$ - quasi-projective (i.e. open in a closed subset of $\mathbb{P}^{n}$)
  - T-stable.

- $\exists$ representation $\rho: T \to \mathbb{P}GL_{n+1}(\mathbb{C})$ & embedding $U_0 \overset{i}{\hookrightarrow} \mathbb{P}^n$

  s.t.
$$\begin{tikzcd} T \times U_0 \arrow[r, hook, "id \times i"] \arrow[d, "\text{given action}"'] \arrow[dr, phantom, "\circlearrowleft"] & T \times \mathbb{P}^n \arrow[d, "\text{action via } \rho"] \\ U_0 \arrow[r, hook, "i"] & \mathbb{P}^n \end{tikzcd}$$

- If $\overline{i(U_0)} = i(U_0)$ then clear $\left(\begin{array}{c}\text{choose some } x \in \mathbb{A}^n \le \mathbb{P}^n \\ \text{and } U = \mathbb{A}^n \cap i(U_0)\end{array}\right)$



$\rightsquigarrow Y := \overline{i(U_0)} \setminus i(U_0)$

choose $f \in \mathbb{C}[x_0, \dots x_n]$ homogeneous s.t. $Y \subseteq \{f = 0\} \not\ni x$

$V := \sum_{t \in T} \mathbb{C} \cdot f(tx) \subseteq \mathbb{C}[x_0, \dots x_n]$

$\dim V < \infty$ since $\deg f(tx)$ is bounded.

$m_y \subseteq \mathbb{C}[x_0, \dots, x_n]$ - ideal of $y \in \mathbb{P}^n$

$\rightsquigarrow W := V \cap \bigcap_{t \in T} m_{tx} \le V$ complete reducibility of Rep(T).

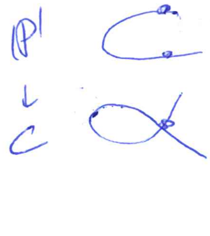
$T \curvearrowright V$, $W \le V$ is T-stable $\Rightarrow \exists F \in V \setminus W$ & $\lambda \in Hom(T, G_m)$

s.t. $F(tx) = \underset{\mathbb{C}^*}{\lambda(t)} \cdot F$

$\Rightarrow x \notin \{F = 0\} \supseteq Y \Rightarrow \mathcal{U} := \overline{i(U_0)} \setminus \{F = 0\} = i(U_0) \setminus \{F = 0\}$ $\circlearrowleft$

Ex. $C \subseteq \mathbb{P}^2$, $C = \{y^2 z = x^2(x+z)\}$ - nodal cubic.

$p = [0;0:1] \in C$ - singular point, $C \setminus \{p\} \simeq G_m$.



One can show that $G_m \hookrightarrow C$ - toric variety,

p has no G_m-invariant affine neighborhoods (the only G_m-inv. neighborhood is C)

~~Thm X-normal toric variety~~

Def. X - normal toric variety, $u \in \mathrm{Hom}(G_m, T)$. We say that $\lim_{t\to 0} \lambda^u(t)$ exists in X if $\exists\, \bar{u} \in \mathrm{Mor}(\mathbb{A}^1, X)$ s.t.

$$\begin{tikzcd} G_m \arrow[r, hook, "u"] \arrow[d, hook] & T \arrow[d, hook] \\ \mathbb{A}^1 \arrow[r, dashed, "\bar{u}"] & X \end{tikzcd}$$

In this case $\lim_{t\to 0} \lambda^u(t) := \bar{u}(0)$.

Rk: $\lim_{t\to 0} \lambda^u(t) = x$ in the above sense iff $\lim_{t\to 0} \lambda^u(t) = x$ in analytic topology.

Prop $\sigma \leq N_\mathbb{R}$ - strongly convex, $u \in N$. $u \in \sigma \iff \lim_{t\to 0} \lambda^u(t)$ exists in U_σ

Pf.

$$\begin{tikzcd} \mathbb{C}[t, t^{-1}] & \mathbb{C}[M] \arrow[l] \\ \mathbb{C}[t] \arrow[u, hook] & \mathbb{C}[S_\sigma] \arrow[l, dashed] \arrow[u, hook] \end{tikzcd}$$

$t^{\langle s,u\rangle} \leftrightarrow x^s$, $\quad S_\sigma = M \cap \sigma^\vee$

$\lim_{t\to 0} \lambda^u(t)$ exists in $U_\sigma \iff \langle s,u\rangle \geq 0 \ \forall s \in S_\sigma$

$\iff u \in \sigma^{\vee\vee} = \sigma.$

Lm. $\tau, \sigma \leq N_\mathbb{R}$ - strongly convex cones, $\tau \subseteq \sigma$ s.t. $U_\tau \subseteq U_\sigma$ - open embedding

$\implies \tau \preceq \sigma$

Pf. <u>Exercise</u> $\tau, \sigma \leq N_\mathbb{R}$ - strongly convex cones, $\tau \subseteq \sigma$.

$\tau \preceq \sigma \iff (u_1, u_2 \in N \cap \sigma,\ u_1 + u_2 \in \tau \implies u_1, u_2 \in \tau)$

$u_1, u_2 \in N \cap \sigma$.

$$\begin{tikzcd} G_m \arrow[r, "\Delta"] \arrow[d, hook] & G_m \times G_m \arrow[r, "u_1 \cdot u_2"] \arrow[d, hook] & T \arrow[d, hook] \\ \mathbb{A}^1 \arrow[rr, dashed] \arrow[d, "\Delta"] & & U_\tau \arrow[d, hook] \\ \mathbb{A}^1 \times \mathbb{A}^1 \arrow[rr, dashed, "\Phi"] & & U_\sigma \end{tikzcd}$$

$u_1 + u_2 \in \tau$

since $u_1, u_2 \in N \cap \sigma$

$\implies \Phi(0,0) \in U_\tau \overset{U_\tau \subseteq U_\sigma \text{ open}}{\implies} \Phi(0,\alpha) \in U_\tau$ for some $\alpha \neq 0$

$\overset{T \curvearrowright U_\tau}{\implies} \Phi(0,1) = u_2(\alpha^{-1}) \cdot \Phi(0,\alpha) \in U_\tau$

$\Phi(0,1) = \lim_{t\to 0} \lambda^{u_1}(t) \implies u_1 \in \tau$ □

Thm X - normal toric variety $\implies X \simeq X_\Sigma$ for some fan Σ.

Pf. Sumihiro + normal affine toric varieties $\implies X = \bigcup U_{\sigma_i}$, $\sigma_i \leq N_\mathbb{R}$, $N := \mathrm{Hom}(T, G_m)^\vee \cong \mathrm{Hom}(G_m, T)$

$U_{\sigma_i} \cap U_{\sigma_j}$ - normal affine toric $\cong U_\tau$, $\tau \leq N_\mathbb{R}$

$\tau = \sigma_i \cap \sigma_j$ - follows from proposition

$\tau \preceq \sigma_i, \sigma_j$ - follows from Lm.

$\Sigma := \{\tau \mid \tau \preceq \sigma_i \text{ for some } i\}$

$\implies X \simeq X_\Sigma$ □

Def. X, X' - varieties, $f: X' \to X$ - *proper* (*universally closed*) if $\forall g: Y \to X$ the morphism $\pi_Y: X' \times_X Y \to Y$ from the diagram
$$\begin{tikzcd} X'\times_X Y \arrow[r] \arrow[d, "\pi_Y"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "g"] & X \end{tikzcd}$$
is *closed*, i.e. $\forall Z \hookrightarrow X' \times_X Y$ closed $\pi_Y(Z) \subseteq Y$ is closed.

X is *proper* if $X \to \operatorname{Spec} \mathbb{C}$ is proper.

Fact X - proper $\iff$ X is compact in analytic topology.

Fact (valuative criterion) X' - irreducible, $W \subseteq X'$ open affine, $X \overset{\#}{=} \bigcup_{\alpha \in A} \mathcal{U}_\alpha$, $f: X' \to X$ s.t. $f(W) \subseteq \bigcap_{\alpha \in A} \mathcal{U}_\alpha$

$f: X' \to X$ - proper iff $\forall \alpha, p, q$ $\exists$ affine open $U \subseteq X'$ s.t. $f(U) \subseteq \mathcal{U}_\alpha$ and $\tilde{p}, \tilde{q}$ s.t.

$$\begin{tikzcd} \mathbb{C}[W] \arrow[rr, "p"] & & \mathbb{C}((t)) \\ & \mathbb{C}[W\cap U] \arrow[ur, "\tilde{p}"] \arrow[ul, phantom, "\subseteq"] & \\ & \mathbb{C}[U] \arrow[u, hook] \arrow[dr, dashed, "\tilde{q}"] & \\ \mathbb{C}[\mathcal{U}_\alpha] \arrow[uuu] \arrow[ur] \arrow[rr, "q"'] & & \mathbb{C}[[t]] \arrow[uuu, hook] \end{tikzcd}$$

[Schematic criterion:
$$\begin{tikzcd} \sim \operatorname{Spec} \mathbb{C}((t)) \arrow[r] \arrow[d, hook] & X' \arrow[d] \\ \sim \operatorname{Spec} \mathbb{C}[[t]] \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$]

Def Σ in $N_\mathbb{R}$ - fan, $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma$ - *support* of Σ.

Prop Σ in $N_\mathbb{R}$, Σ' in $N'_\mathbb{R}$, $\varphi: \Sigma' \to \Sigma$. $f_\varphi: X_{\Sigma'} \to X_\Sigma$ - proper $\iff \varphi^{-1}(|\Sigma|) = |\Sigma'|$

Pf. "$\Rightarrow$" Pick $v \in N'$ s.t. $\varphi(v) \in \sigma \subseteq |\Sigma| \rightsquigarrow \lim_{t\to 0} \lambda^{\varphi(v)}$ exists in X_Σ and it suffices to show that $\lim_{t \to 0} \lambda^v(t)$ exists in $X_{\Sigma'}$

$$\begin{tikzcd} G_m \arrow[r, "f_v"] \arrow[d, hook] \arrow[dr, "f_{\varphi(v)}"] & X_{\Sigma'} \arrow[d, "f_\varphi"] \\ \mathbb{A}^1 \arrow[ur, dashed, "?"] \arrow[r, dashed, "\overline{f}_{\varphi(v)}"'] & X_\Sigma \end{tikzcd}$$

<u>Exercise</u>: prove it. Either use valuative criterion, or consider $f_\varphi(\overline{f_v(G_m)}) \supseteq \overline{f}_{\varphi(v)}(\mathbb{A}^1)$
f_φ is closed.

"$\Leftarrow$"
$$\begin{tikzcd} \mathbb{C}[U_{\{0\}}] \arrow[r, "p"] & \mathbb{C}((t)) \arrow[r, "\mathrm{ord}"] & \mathbb{Z} \\ \mathbb{C}[U_\sigma] \arrow[u] \arrow[r, "q"] & \mathbb{C}[[t]] \arrow[u, hook] & \end{tikzcd}$$

$\operatorname{ord}(\sum a_i t^i) = m \in \mathbb{Z}$, where $a_m \neq 0$, $a_i = 0$ for $m > i$.

$\rightsquigarrow$ for $s' \in (N')^\vee$ have $x^{s'} \in \mathbb{C}[U_{\{0\}}]$

$\rightsquigarrow (N')^\vee \longrightarrow \mathbb{Z}$
$s \longmapsto \operatorname{ord}(p(x^s))$.

$\rightsquigarrow v \in N'$ s.t. $\langle s', v\rangle = \operatorname{ord}(p(x^{s'}))$ $\forall s' \in (N')^\vee$

for $s \in N^\vee \cap \sigma^\vee$ $\langle s, \varphi(v)\rangle = \langle \varphi^\vee(s), v\rangle = \operatorname{ord}(p(x^{\varphi^\vee(s)})) =$
$= \operatorname{ord}(q(x^s)) \geq 0$

$\Rightarrow \varphi(v) \in (N^\vee \cap \sigma^\vee)^\vee = N \cap \sigma \implies \exists \sigma' \in \Sigma'$ s.t. $v \in \sigma'$ & $\varphi(\sigma') \subseteq \sigma$
$\varphi^{-1}|\Sigma| = |\Sigma'|$

$\forall s' \in (\sigma')^\vee$ $\operatorname{ord}(p(x^{s'})) = \langle s', v\rangle \geq 0 \Rightarrow$
$$\begin{tikzcd} & \mathbb{C}[U_{\{0\}}] \arrow[r, "p"] & \mathbb{C}((t)) \\ \mathbb{C}[U_{\sigma'}] \arrow[ur, hook] \arrow[rr, dashed, "\tilde{q}"] & & \\ & \mathbb{C}[U_\sigma] \arrow[ul] \arrow[uu] \arrow[r, "q"] & \mathbb{C}[[t]] \arrow[uu] \end{tikzcd}$$

Def Σ, Σ' - fans in $N_{\mathbb{R}}$. Σ' refines Σ if 1) $\forall \sigma' \in \Sigma'\ \exists \sigma \in \Sigma$ s.t. $\sigma' \subseteq \sigma$
2) $|\Sigma| = |\Sigma'|$

Cor. Σ, Σ' - fans in $N_{\mathbb{R}}$, Σ' refines Σ $\Rightarrow$ $X_{\Sigma'} \xrightarrow{id} X_{\Sigma}$ - proper birational morphism
iso on an open subset.

Lm Σ - fan in $N_{\mathbb{R}}$, $\sigma \in \Sigma$ s.t. $\sigma = \mathrm{Cone}(e_1, \dots e_n)$,
where $e_1, \dots e_n$ - basis of N. Put $e_0 := e_1 + \dots + e_n$, $\sigma_I := \mathrm{Cone}(\{e_i\}_{i \in I})$ for $I \subseteq \{0, \dots n\}$.
$\Sigma' := (\Sigma \setminus \{\sigma\}) \cup \{\sigma_I\}_{\substack{I \subseteq \{0, \dots n\} \\ I \not\supseteq \{1, \dots n\}}}$ Then 1) Σ' is a refinement of Σ
2) $X_{\Sigma'} \to X_{\Sigma}$ is the blow-up of X_{Σ} at the point $x \in U_\sigma \subseteq X_\Sigma$
 $x \leftrightarrow 0 \in \mathbb{A}^n$, $U_\sigma \cong \mathbb{A}^n$

Pf. (1) is clear

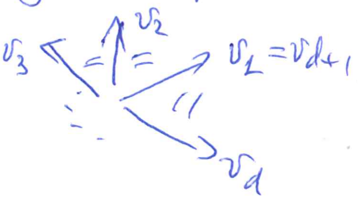
(2) $X_{\Sigma'} = \bigcup_{\tau \in (\Sigma \setminus \{\sigma\}) \cup \Sigma^*(\sigma)} U_\tau \longrightarrow \bigcup_{\tau \in \Sigma} U_\tau = X_\Sigma$

- iso over every U_τ, $\tau \neq \sigma$. Over U_σ we have $\bigcup_{\tau \in \Sigma^*(\sigma)} U_\tau \to U_\sigma$ - this is $Bl_0 \mathbb{A}^n$ by a preceding exercise.

Ex.  $\Rightarrow \mathcal{H}_1 \simeq Bl_{(0)} \mathbb{P}^2$

Thm X_Σ - smooth proper surface $\Rightarrow$ X_Σ can be obtained from $\mathbb{P}^2$ or one of $\mathcal{H}_r$, $r \neq 1$, via a sequence of blow-ups at T-invariant pts.

Rk. Similar hold for any smooth proper rational surface; but for toric surfaces intermediate surfaces given by blow-ups are toric.

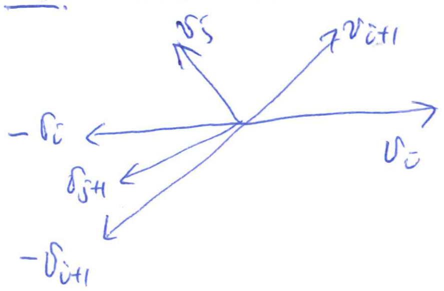
X_Σ - smooth proper surface $\Rightarrow$ $\Sigma \rightsquigarrow$ 

Necessary & sufficient conditions:
1) v_i, v_{i+1} - basis of $\mathbb{Z}^2$ $\forall i = 1, \dots d$.
2) the angle between v_i, v_{i+1} less $180°$.

$\Rightarrow v_{i+1} = \pm v_{i-1} + a_i v_i$ $\Rightarrow$ $a_i v_i = v_{i+1} + v_{i-1}$ for some $a_i \in \mathbb{Z}$.
 -1, because of the angle i

$d=3$ may assume $v_1 = e_1$, $v_2 = e_2$ $\Rightarrow a_3 v_3 = e_1 + e_2 \Rightarrow a_3 = -1 \Rightarrow \mathbb{P}^2$

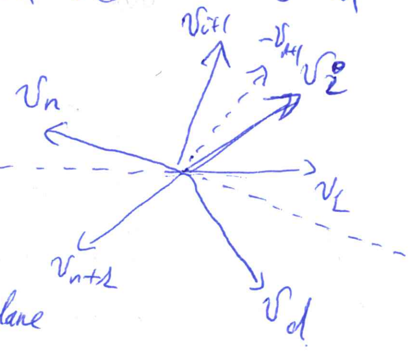
Lm. There could not be such configuration:



Pf. Suppose such exists, may assume $v_i = e_1$, $v_{i+1} = e_2$. $v_j = ae_1 + be_2$, $v_{j+1} = ce_1 + de_2$. It follows from configuration that $a, c, d < 0$, $b > 0$. $\Rightarrow$
$\Rightarrow \underset{\geq 1}{ad} - \underset{\leq -2}{bc} \geq 2 \Rightarrow v_j, v_{j+1}$ is not a basis?!

Lm $d \geq 4 \Rightarrow \exists i, j$ s.t. $v_i = -v_j$.

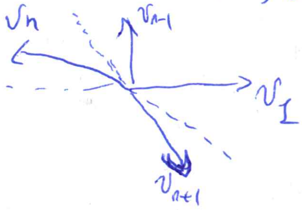
Pf Suppose not, consider the longest consec. chain of vectors in the same halfplane, wlog $v_1, \dots, v_n$.



$-v_{n+1}$ is between (v_i, v_{i+1}) for some $1 \leq i \leq n-1$
(if is less than v_n, otherwise enlarge)

forbidden configuration of (v_i, v_{i+1}) & (v_n, v_{n+1})

$\Rightarrow i = n-1$. $\Rightarrow$

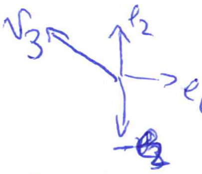


$\Rightarrow$ all except v_n are in the same halfplane

$\Rightarrow n+1 = d$

$\Rightarrow$ forbidden configuration for (v_d, v_1) & (v_{n-1}, v_n) $\square$

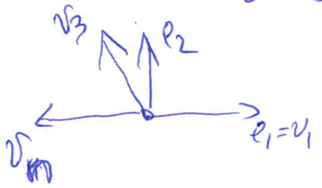
$d=4$ $\rightsquigarrow$



$a e_2 = v_3 + e_1 \Rightarrow v_3 = -e_1 + a e_2 \Rightarrow X_\Sigma \simeq \mathcal{H}_{-a}$

Lm. $d \geq 5 \Rightarrow \exists i$ s.t. $v_i = v_{i+1} + v_{i-1}$.

Pf.



wlog $n \geq 4$ since $d \geq 5$.

$v_j = b_j e_1 + c_j e_2$; $A_j := c_j - b_j \geq 1$ for $2 \leq i \leq n$.

$A_2 = 1$, $A_3 \geq 2$, $A_n = 1$

$\Rightarrow \exists\ 3 \leq i \leq n-1$ s.t. $A_i > A_{i+1}$, $A_i \geq A_{i-1}$ (e.g. A_i is the last maximum)

$$a_i (b_i e_1 + c_i e_2) = (b_{i-1} + b_{i+1}) e_1 + (c_{i-1} + c_{i+1}) e_2$$

$$\Rightarrow \begin{cases} a_i b_i = b_{i-1} + b_{i+1} \\ a_i c_i = c_{i-1} + c_{i+1} \end{cases} \Rightarrow a_i \cdot A_i = A_{i-1} + A_{i+1} \xRightarrow{A_i > A_{i+1} \geq 1,\ A_i \geq A_{i-1} \geq 1} a_i = 1$$

$\Rightarrow v_i = v_{i-1} + v_{i+1}$ $\square$

Pf (of the Thm on minimal toric surfaces):

Induction on d. $d = 3, 4$ – know. If $d \geq 5 \Rightarrow \exists i$ s.t. $v_i = v_{i-1} + v_{i+1}$

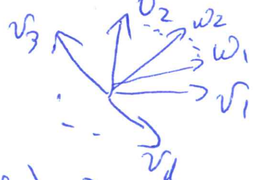
$\rightsquigarrow$ consider Σ' given by all v_j except $v_i \Rightarrow X_\Sigma \to X_{\Sigma'}$ – blow-up,

d for Σ' is less $\Rightarrow$ induction $\square$.

Toric resolution of singularities

(x) **Thm.** Σ in $\mathbb{R}^2$ – fan $\Rightarrow \exists$ refinement Σ' of Σ s.t. Σ' is smooth (consists of smooth cones)

Pf. Σ is generated by $v_1, \dots, v_d$

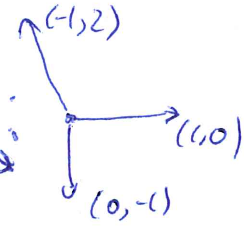


It is sufficient to find

$v_1 = w_0, w_1, w_2, \dots, w_k = v_2$ in Cone(v_1, v_2) in such order, s.t. $\{w_i, w_{i+1}\}$ – basis of $\mathbb{Z}^2$ $\forall i$.

(x) Ex: Resolve

a)



b)

Wlog $\overset{w_0 := v_1}{\cancel{\quad}}$. Choose arbitrary w_1 s.t. w_0, w_1 - basis of $\mathbb{Z}^2$.

$v_2 = -k_1 w_0 + m_1 w_1$. Changing $\begin{matrix} w_1 \to -w_1 \\ w_1 \to w_1 + a w_0 \end{matrix}$ we may assume

that $m_1 \geq 0$, $0 \leq k_1 < m_1$; also $(k_1, m_1) = 1$ since v_2 is a prime generator of the ray.

If $k_1 = 0$ then there is nothing to prove - v_2, w_0 - basis.

Otherwise choose w_2 s.t. $\{w_1, w_2\}$ - basis of $\mathbb{Z}^2$ and

$v_2 = -k_2 w_1 + m_2 w_2$ with $m_2 \geq 0$, $0 \leq k_2 < m_2$.

Change of the basis is given by $\begin{pmatrix} [\frac{m_1}{k_1}]+1 & 1 \\ -1 & 0 \end{pmatrix} \rightsquigarrow m_2 = k_1 < m_1$.

If $k_2 = 0$ then we are done, otherwise choose w_3 s.t. w_2, w_3 - basis...
m_i decreases $\Rightarrow$ terminates $\square$.

Rk. $\frac{m_1}{k_1} = a_1 - \frac{k_2}{m_2} = a_1 - \frac{1}{\frac{m_2}{k_2}} = a_1 - \frac{1}{a_2 - \frac{1}{\ddots}}$ with $a_i \geq 2$, $a_i \in \mathbb{N}$.

- Hirzebruch-Jung continued fraction of m_1/k_1.

Cor Σ in $\mathbb{R}^2$, ~~[illegible]~~ $\Rightarrow \exists$ proper birational toric morphism $X_{\Sigma'} \to X_\Sigma$ with $X_{\Sigma'}$ being smooth.

Thm X - toric variety $\Rightarrow \exists$ fan Σ and a toric proper birational morphism $X_\Sigma \to X$ s.t. X_Σ is smooth.

Pf: 1) $\tilde{X} \xrightarrow{\text{proper birational}} X$ - normalization

$$\begin{tikzcd} T \times \tilde{X} \arrow[r, dashed] \arrow[d] & \tilde{X} \arrow[d] \\ T \times X \arrow[r] & X \end{tikzcd}$$

normal, π dominant

$\rightsquigarrow \tilde{X}$ - toric, i.e. $\tilde{X} = X_{\Sigma_1}$.

Construction Σ in $N_\mathbb{R}$ - fan. $v \in |\Sigma| \cap N \rightsquigarrow \Sigma^*(v) = \{\sigma\}_{\sigma \not\ni v} \cup \{\text{Cone}(\tau, v)\}_{\substack{v \notin \tau \in \Sigma \\ \{0\} \cup \tau \leq \sigma \in \Sigma}}$

- star (stellar) subdivision of Σ.

Exercise: $\dim = \dim \tau + 1$

Claim: $\Sigma^*(v)$ - cone which is a refinement of Σ.

Pf: straightforward but lengthy (Exercise)

2) Taking star subdivisions of Σ_1 at edges repeatedly one obtains a simplicial fan (i.e. every $\sigma \in \Sigma_1$ is gen. by $\dim \sigma$ vectors)

edges do not increase $\Rightarrow$ process terminates on Σ_2, the fan Σ_2 is simplicial.

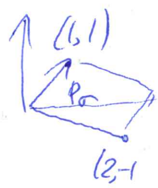
Suppose not, $\sigma \in \Sigma_2$ is not simpl. of minimal dim. $\Rightarrow \sigma = \text{Cone}(v_1, \dots v_k)$, $k > \dim \sigma$.

$\Sigma_2^*(v_1) = \Sigma_2 \Rightarrow \sigma = \text{Cone}(v_1, \tau)$ for some $\tau \in \Sigma_2$, $v_1 \notin \tau$. $\dim \text{Cone}(v_1, \tau) = \dim \tau + 1 \Rightarrow$

$\Rightarrow \dim \tau = \dim \sigma - 1 \Rightarrow \tau$ - simplicial $\Rightarrow \tau = \text{Cone}(w_1, \dots w_r)$, $r = \dim \sigma - 1$, $\sigma = \text{Cone}(v_1, \tau) = \text{Cone}(v_1, w_1, \dots, w_r)$

3) Σ_2 - simplicial, $\sigma\in\Sigma_2$, $u_1,\dots u_k$ - minimal generators of σ,
$\mathrm{mult}(\sigma) := |N_\sigma / \mathbb{Z}u_1+\dots+\mathbb{Z}u_k| \in \mathbb{N}$. $\mathrm{mult}(\sigma)=1 \iff \sigma$ - smooth.

Claim: $\mathrm{mult}(\sigma) = |P_\sigma\cap N|$, where $P_\sigma = \{\sum_{i=1}^k \alpha_i u_i \mid 0\le\alpha_i<1\}$.



Pf: $P_\sigma\cap N \to N\cap\sigma \to N\cap\sigma / \mathbb{Z}u_1+\dots+\mathbb{Z}u_k$ - bijection.

Cor: $\tau\le\sigma \Rightarrow \mathrm{mult}(\sigma)\ge\mathrm{mult}(\tau)$.

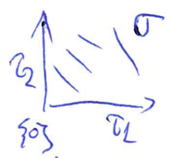
Claim. $\mathrm{mult}(\sigma) = |\det A_u|$, where A_u - matrix of u_i in some basis for N_σ
Pf: Exercise.

Now suppose $\sigma\in\Sigma_2$, $\mathrm{mult}(\sigma)>1$. Pick $v\in P_\sigma\cap N$, then for $v\notin\tau\le\sigma$ we have $\mathrm{mult}(\mathrm{Cone}(v,\tau)) < \mathrm{mult}\,\sigma$. Indeed, $\sigma=\mathrm{Cone}(\underset{\text{min-gen.}}{u_1,\dots u_k})$, & $v=\sum_{i=1}^k \alpha_i u_i$, $0\le\alpha_i<1$
wlog $u_k\notin\tau$ & $\alpha_k\neq 0 \Rightarrow \mathrm{Cone}(\tau,v)\le\mathrm{Cone}(u_1,\dots,u_{k-1},v)$
$\Rightarrow \mathrm{mult}(\mathrm{Cone}(\tau,v)) \le \mathrm{mult}(\mathrm{Cone}(u_1,\dots u_{k-1},v))$.
$\mathrm{mult}(\sigma) = |\det(A_u)| > \alpha_k\cdot|\det A_u| = |\det(A_{u_1,\dots u_{k-1},v})| = \mathrm{mult}(\mathrm{Cone}(u_1,\dots u_{k-1},v))$

For a simplicial fan Σ put $\mathrm{mult}(\Sigma) := \max_{\sigma\in\Sigma}(\mathrm{mult}(\sigma))$,
$\mathrm{nmult}(\Sigma) := |\{\sigma\in\Sigma \mid \mathrm{mult}(\sigma)=\mathrm{mult}(\Sigma)\}|$

suppose $\mathrm{mult}(\Sigma_2)>1$, pick $\sigma\in\Sigma_2$ s.t. $\mathrm{mult}(\Sigma_2)=\mathrm{mult}(\sigma)$, choose $0\neq v\in P_\sigma\cap N$. Then either $\mathrm{mult}(\Sigma_2^*(v)) < \mathrm{mult}(\Sigma_2)$
or $\mathrm{mult}(\Sigma_2^*(v)) = \mathrm{mult}(\Sigma_2)$ & $\mathrm{nmult}(\Sigma_2^*(v)) < \mathrm{nmult}(\Sigma_2)$.
$\rightsquigarrow$ induction $\square$

Torus orbits.

Ex: 

$\Sigma=\{\sigma,\tau_1,\tau_2,\{0\}\} \rightsquigarrow X_\Sigma \simeq U_\sigma \simeq \mathbb{A}^2$. Orbits for $G_m^2 \curvearrowright \mathbb{A}^2$:

$\{0\}$, $\{0\}\times G_m$, $G_m\times\{0\}$, $G_m\times G_m$
$\updownarrow$ σ, τ_1, τ_2, $\{0\}$

Def. $\sigma\subseteq N_{\mathbb{R}}$ - strongly convex, $S_\sigma := \sigma^\vee\cap M$, $M:=N^\vee\simeq\mathrm{Hom}(T,G_m)$
$U_\sigma = \mathrm{Spec}\,\mathbb{C}[S_\sigma]$

$\rightsquigarrow \{x\in U_\sigma\} \xleftrightarrow{1:1} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[S_\sigma],\mathbb{C}) \xleftrightarrow{1:1} \mathrm{Hom}_{\text{monoids}}(S_\sigma,(\mathbb{C},\cdot))$
$f \longmapsto \varphi_f$, $\varphi_f(m) = f(x^m)$
$f_\varphi(x^m) := \varphi(m)$, $\mathbb{C}$-linear. $\quad f_\varphi \longleftarrow\!\shortmid \varphi$

In particular, if $\sigma=\{0\} \rightsquigarrow \underset{U_\sigma=T}{S_\sigma = M} \rightsquigarrow \{x\in T\} \xleftrightarrow{1:1} \mathrm{Hom}_{\text{monoids}}(M,(\mathbb{C},\cdot)) \simeq \mathrm{Hom}_{\mathbb{Z}}(M,\mathbb{C}^*)$

The action: $T\times U_\sigma \to U_\sigma \rightsquigarrow \mathrm{Hom}_{\text{mon}}(M,\mathbb{C})\times\mathrm{Hom}(S_\sigma,\mathbb{C}) \to \mathrm{Hom}(S_\sigma,\mathbb{C})$, $(\delta,\gamma)\mapsto(m\mapsto\delta(m)\cdot\gamma(m))$
- Exercise

Def. $\sigma \subseteq N_{\mathbb{R}}$ – strongly convex cone, $\sigma^{\perp} := \{m \in M_{\mathbb{R}} \mid \langle m,u\rangle = 0 \ \forall u \in \sigma\}$

$\sigma^{\perp}$ the minimal face of $\sigma^{\vee}$ $\cap$ $\sigma^{\vee} := \{m \in M_{\mathbb{R}} \mid \langle m,u\rangle \geq 0 \ \forall u \in \sigma\}$

$\mathcal{O}(\sigma) := \{\gamma : S_\sigma \to \mathbb{C} \mid \gamma^{-1}(\mathbb{C}^*) = \sigma^{\perp} \cap M\} \subseteq U_\sigma$

Lm 1) $\mathcal{O}(\sigma) \subseteq U_\sigma$ is a T-orbit

2) $\mathcal{O}(\sigma) \simeq T_{\sigma^\perp \cap M}$, where $T_{\sigma^\perp \cap M}$ is the torus with the lattice of characters $\sigma^\perp \cap M \subseteq M$ as T-sets. $(\Rightarrow T \twoheadrightarrow T_{\sigma^\perp \cap M})$

Pf. $\gamma(m) \neq 0 \Leftrightarrow \alpha(m)\cdot\gamma(m) \neq 0$, where $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) \simeq \mathrm{Hom}_{mon}(M,(\mathbb{C},\cdot)) \simeq \{x \in T\}$

$\Rightarrow \mathcal{O}(\sigma)$ is T-stable

$\gamma \in \mathcal{O}(\sigma) \Rightarrow \gamma|_{\sigma^\perp \cap M} : \sigma^\perp \cap M \to \mathbb{C}^*$ – homo-sm of abelian groups

$\rightsquigarrow \mathcal{O}(\sigma) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, \mathbb{C}^*) \simeq \{x \in T_{\sigma^\perp \cap M}\}$

$\gamma \longmapsto \gamma|_{\sigma^\perp \cap M}$

$m \mapsto \begin{cases} \gamma'(m), & m \in \sigma^\perp \cap M \\ 0, & m \in S_\sigma \setminus (\sigma^\perp \cap M) \end{cases} \longleftarrow\!\shortmid \gamma'$ $\Rightarrow \mathcal{O}(\sigma) \simeq \{x \in T_{\sigma^\perp \cap M}\}$ – bijection.

It is $T \simeq \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ – equivariant $\square$

Thm. Σ – fan in $N_{\mathbb{R}}$. Then

1) $\mathcal{O} : \Sigma \longrightarrow \{T\text{-orbits in } X_\Sigma\}$

$\sigma \longmapsto \mathcal{O}(\sigma) \subseteq U_\sigma \subseteq X_\Sigma$

2) $\dim \mathcal{O}(\sigma) = n - \dim \sigma$

3) $\sigma \in \Sigma \Rightarrow U_\sigma = \bigcup_{\tau \preceq \sigma} \mathcal{O}(\tau)$

4) $\overline{\mathcal{O}(\tau)} = \bigcup_{\tau \preceq \sigma} \mathcal{O}(\sigma)$

Pf. Pick $\mathcal{O} \subseteq X_\Sigma$ a T-orbit. Since $X_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$ then $\mathcal{O} \subseteq U_\sigma$ for some σ.

Claim: $\mathcal{O} = \mathcal{O}(\tau) \subseteq U_\tau \subseteq U_\sigma$ for some $\tau \preceq \sigma$

Pick $\gamma \in \mathcal{O}$; $\gamma : S_\sigma \to \mathbb{C}$. ~~$\forall\, \alpha, \alpha' \in \mathrm{Cone}(\gamma^{-1}(\mathbb{C}^*)) \cap M$ s.t. $\alpha+\alpha' \in$~~

$\forall\, m, m' \in \sigma^\vee \cap M$ ($= S_\sigma$) s.t. $m+m' \in \mathrm{Cone}(\gamma^{-1}(\mathbb{C}^*)) \cap M$ ($= \gamma^{-1}(\mathbb{C}^*)$ $\Leftarrow$ $\gamma^{-1}(\mathbb{C}^*)$ is saturated in S_σ) $\xRightarrow{\gamma(m)\cdot\gamma(m') = \gamma(m+m') \neq 0 \Rightarrow} m, m' \in \gamma^{-1}(\mathbb{C}^*)$

Fact: $\tau \subseteq \sigma \subseteq M_{\mathbb{R}}$ – cones, suppose $(m, m' \in \sigma \cap M,\ m+m' \in \tau \Rightarrow m, m' \in \tau)$. Then $\tau \preceq \sigma$.

$\Rightarrow \mathrm{Cone}(\gamma^{-1}(\mathbb{C}^*)) \preceq \sigma^\vee \Rightarrow \mathrm{Cone}(\gamma^{-1}(\mathbb{C}^*)) = \tau^\perp \cap \sigma^\vee$ for some $\tau \preceq \sigma$

$\Rightarrow \gamma^{-1}(\mathbb{C}^*) = \tau^\perp \cap \sigma^\vee \cap M \Rightarrow \gamma \in \mathcal{O}(\tau) \xRightarrow[\mathcal{O}(\sigma) \text{ is a } T\text{-orbit}]{} \mathcal{O} = \mathcal{O}(\tau)$.

$\Rightarrow \Theta : \Sigma \to \{T\text{-orbits in } X_\Sigma\}$ is surjective.
$\sigma \mapsto \Theta(\sigma)$

$\dim \Theta(\sigma) = \dim T_{\sigma^\perp \cap M} = \dim \sigma^\perp = n - \dim \sigma. \Rightarrow 2)$

Suppose $\underset{\subseteq\, U_{\sigma_1}}{\Theta(\sigma_1)} = \underset{\subseteq\, U_{\sigma_2}}{\Theta(\sigma_2)}$ $\Rightarrow \Theta(\sigma_1) = \Theta(\sigma_2) \subseteq U_{\sigma_1 \cap \sigma_2} \doteq U_{\sigma_1} \cap U_{\sigma_2}$

$\Rightarrow \Theta(\sigma_1) = \Theta(\sigma_2) = \Theta(\tau)$ for some $\tau \leq \sigma_1 \cap \sigma_2 \leq \sigma_1, \sigma_2$

$\Rightarrow \left.\begin{array}{l} \dim \Theta(\tau) = n - \dim \tau \\ \dim \Theta(\sigma_1) = n - \dim \sigma_1 \\ \dim(\Theta(\sigma_2)) = n - \dim \sigma_2 \end{array}\right\} \Rightarrow \dim \tau = \dim \sigma_1 = \dim \sigma_2 \Rightarrow \tau = \sigma_1 \cap \sigma_2 = \sigma_1 = \sigma_2.$

$\Rightarrow 1)$.

3): $\Theta(\tau) \subseteq U_\tau \subseteq U_\sigma \overset{\text{for } \tau \leq \sigma}{\Rightarrow} U_\sigma \supseteq \bigcup_{\tau \leq \sigma} \Theta(\tau)$

Pick $\Theta \subseteq U_\sigma$, $\Rightarrow \Theta = \Theta(\tau)$ for $\tau \leq \sigma$ $\Rightarrow U_\sigma = \bigcup_{\tau \leq \sigma} \Theta(\tau)$.

4). $\overline{\Theta(\tau)}$ is T-invariant $\Rightarrow \overline{\Theta(\tau)} = \bigcup_{\sigma \in ?} \Theta(\sigma)$

Pick $\Theta(\sigma) \subseteq \overline{\Theta(\tau)} \Rightarrow \overline{\Theta(\tau)} \cap U_\sigma \neq \emptyset \underset{U_\sigma\text{-open}}{\Rightarrow} \Theta(\tau) \cap U_\sigma \neq \emptyset \Rightarrow \Theta(\tau) \subseteq U_\sigma \Rightarrow \tau \leq \sigma$
$\Rightarrow \overline{\Theta(\tau)} \subseteq \bigcup_{\tau \leq \sigma} \Theta(\sigma)$.

Assume $\tau \leq \sigma$. It suffices to show that
$\overline{\Theta(\tau)} \cap \Theta(\sigma) \neq \emptyset$. ($\overset{T\text{-inv.}}{\Rightarrow} \overline{\Theta(\tau)} \supseteq \Theta(\sigma)$)

Pick
- $\gamma \in \Theta(\tau)$, $\gamma : S_\tau \to \mathbb{C}$
$\gamma(m) = \begin{cases} 1, & m \in \tau^\perp \cap M \\ 0, & m \notin \tau^\perp \cap M \end{cases}$
- $u \in \sigma \cap N$ s.t. u is in the relative interior of σ, i.e.
$\langle m, u \rangle > 0 \ \forall m \in \sigma^\vee \setminus \sigma^\perp$

$\rightsquigarrow \ \mathbb{G}_m \xrightarrow{\varphi} \Theta(\tau) \subseteq U_\tau$. Claim: φ extends to $\overline{\varphi} : \mathbb{A}^1 \to U_\sigma$ & $\overline{\varphi}(0) \in \Theta(\sigma)$
$t \mapsto \underset{\in T}{u(t)} \cdot \underset{\in \Theta(\tau)}{\gamma}$

Pf.
$$\begin{array}{ccccc} & & \mathbb{C}[t, t^{-1}] & \xleftarrow{\ t^{\langle m,u\rangle}\gamma(m) \leftarrow\!\shortmid\, \chi^m\ } & \mathbb{C}[S_\tau] \\ & & \uparrow & & \uparrow \\ \mathbb{C} & \xleftarrow{t=0} & \mathbb{C}[t] & \xleftarrow{\ \delta\ } & \mathbb{C}[S_\sigma] \end{array}$$

$u \in \sigma \Rightarrow \langle m, u \rangle \geq 0 \ \forall m \in S_\sigma \Rightarrow$ extends

$\Rightarrow \overline{\varphi}(0) \in \overline{\Theta(\tau)} \cap \Theta(\sigma)$ $\square$.

u is in rel. interior $\Rightarrow \langle m, u \rangle = 0 \Leftrightarrow m \in \sigma^\perp$

$\overset{t=0}{\rightsquigarrow} \ \delta|_{t=0}(\chi^m) = \begin{cases} \gamma(m) = 1, & m \in \sigma^\perp \\ 0, & m \notin \sigma^\perp \end{cases}$

$\Rightarrow \overline{\varphi}(0) \in \Theta(\sigma)$ $\square$.

Def (reminder) $f: X_{\Sigma_1} \to X_{\Sigma_2}$ – toric morphism, if $f(T_1) \subseteq T_2$ and $f|_{T_1}: T_1 \to T_2$ – homo-sm. $\varphi: N_1 \to N_2$ is a morphism of fans if it is a homomorphism and $\forall \sigma_1 \in \Sigma_1\ \exists \sigma_2 \in \Sigma_2$ s.t. $\varphi_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$.

$\varphi: \Sigma_1 \to \Sigma_2$ – morphism of fans $\rightsquigarrow$ morphisms $U_{\sigma_1} \to U_{\sigma_2}$ $\rightsquigarrow$

$\rightsquigarrow$ toric morphism $f_\varphi : X_{\Sigma_1} \to X_{\Sigma_2}$.

Proposition. $f: X_{\Sigma_1} \to X_{\Sigma_2}$ – toric morphism. $\Rightarrow \exists \varphi: \Sigma_1 \to \Sigma_2$ s.t. $f = f_\varphi$.

Pf. $N_1 = Hom(G_m, T_1)$, $N_2 = Hom(G_m, T_2)$ $\to$ f induces a homo-sm $\varphi: N_1 \to N_2$.

f – toric $\Rightarrow$ f is equivariant, i.e.

$$\begin{array}{ccc} T_1 \times X_{\Sigma_1} & \longrightarrow & X_{\Sigma_1} \\ \downarrow f|_{T_1} \times f & (*) & \downarrow f \\ T_2 \times X_{\Sigma_2} & \longrightarrow & X_{\Sigma_2} \end{array}$$

commutes

((*) – because $\rceil = L$, one a dense open $T_1 \times T_1$, and in general $f = g$ on a closed)

pick $\sigma_1 \in \Sigma_1 \rightsquigarrow O(\sigma_1) \subset X_{\Sigma_1}$ – T_1-orbit $\Rightarrow \exists\, O(\sigma_2) \supseteq f(O(\sigma_1))$, $\sigma_2 \in \Sigma_2$

if $\tau_1 \preceq \sigma_1 \Rightarrow \exists \tau_2 \in \Sigma_2$ s.t. $f(O(\tau_1)) \subseteq O(\tau_2)$

So $\overline{O(\tau_1)} \supseteq O(\sigma_1) \Rightarrow f(\overline{O(\tau_1)}) \supseteq f(O(\sigma_1)) \neq \emptyset$

$$\begin{array}{ccc} \cap\!\!\mid & & \cap\!\!\mid \\ \overline{O(\tau_2)} & \subseteq & O(\sigma_2) \end{array} \Rightarrow \tau_2 \preceq \sigma_2$$

$\Rightarrow f(U_{\sigma_1}) = f\Big(\bigcup\limits_{\tau_1 \preceq \sigma_1} O(\tau_1)\Big) \subseteq \bigcup\limits_{\tau_2 \preceq \sigma_2} O(\tau_2) = U_{\sigma_2}$

$\Rightarrow f|_{U_{\sigma_1}}: U_{\sigma_1} \to U_{\sigma_2}$ – toric $\Rightarrow \varphi_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$

$\Rightarrow \varphi$ – morphism of fans, $f|_{U_\sigma} = f_\varphi|_{U_\sigma}\ \forall \sigma \in \Sigma \Rightarrow \square$